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VOL. XV UNIVERSITY, LA, JANUARY, 1941. No. 4

Entered as second-class matter at University, Louisiana.

Published monthly excepting June, July, August, September, by LOUISIANA STATE UNIVERSITY, Vols. 1-8 Published as MATHEMATICS NEWS LETTER.

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Some Characteristics of a Good Review of an Elementary Mathematics Textbook

We have no desire to set up fixed standards for reviews that are to appear in the National Mathematics Magazine; we only wish to express a few ideas which seem to be fairly well accepted, though they are frequently ignored.

Relative to reviews of textbooks that are used in large volume, as for freshmen and sophomore mathematics courses,—trigonometry, college algebra, analytical geometry, and calculus,—it seems clear that a reviewer of such a book should report correctly upon at least *three things*: *the content of the text*; *the exposition*, including the worked examples and diagrams of the text; and *the quality of the problem list*.

Unless the reviewer indicates the general nature of the content of the next and mentions its variations from the content of other books on the same subject and in the same role, a teacher could not learn from the reviewer whether the text would likely be suitable for the courses on the subject that he gives.

The main difference between a good text and a bad one is very frequently that the good one contains fuller, clearer explanation and more well-worked examples than does the bad one.

A good problem list is one of the best assets of a new text of the type under consideration. The exposition in many available textbooks on trigonometry, college algebra, analytical geometry, and calculus is *excellent*: in looking through several available books on each of these subjects, one is apt to feel that there is no great need of better exposition. But time surely develops need of new, thought-provoking problems.

In singling out the three topics mentioned above, we of course do not wish to imply that it is out of place to comment on the format, the correctness of answers, and several other features that are apt to be of some interest to many readers. Our point is that those three topics are of outstanding importance, and that we do not believe they should ever be ignored.

H. A. Simmons.
Northwestern University.

Algebraic Surfaces Invariant Under the Symmetric G_{120} with Special Reference to Quintics and Sextics

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I. *Introduction*

The purpose of this paper is to study the general surface of order n invariant under the symmetric group of order 120 and to study, as particular cases, the surfaces of orders five and six.

All the substitutions of the symmetric group of order 120 (we shall refer to the group as the G_{120} hereafter) can be represented as follows:*

$$\begin{pmatrix} X_i\ X_j\ X_k\ X_l\ X_m \\ X_1\ X_2\ X_3\ X_4\ X_5 \end{pmatrix}$$

where $X_i\ X_j\ X_k\ X_l\ X_m$ is some permutation of $X_1\ X_2\ X_3\ X_4\ X_5$. In order to study surfaces of ordinary space which are invariant under the G_{120} we assume $X_1+X_2+X_3+X_4+X_5=0$.

Any four of the planes $\pi_i(X_i=0)$ can be considered as faces of the coordinate tetrahedron and the fifth as the unit plane.

The ten edges and the ten vertices of the pentalateral formed by the five planes π_i will be designated by s_{ij} and v_{ij}, respectively. For example, s_{12} is the intersection of π_1, and π_2, and v_{12} is the intersection of π_3, π_4, and π_5. The edge s_{ij} and the vertex v_{ij} determine a plane δ_{ij}, called a diagonal plane. Any four faces of the pentalateral intersects the fifth in a quadrilateral, the four sides and six vertices of which are edges and vertices of the pentalateral. Each quadrilateral has three diagonal lines and three diagonal points. The diagonal line that joins v_{ij} and v_{kl} we denote by d_{ijkl}, and the diagonal point which is the intersection of d_{ikjl} and d_{ijkl} is designated by D_{ijkl}.

The polar reciprocals of the five faces π_i, the ten vertices v_{ij}, the ten edges s_{ij}, the fifteen diagonal lines d_{ijkl}, the fifteen diagonal points D_{ijkl}, the ten diagonal planes δ_{ij} of the pentalateral with respect to the quadric $X_1^2+X_2^2+X_3^2+X_4^2+X_5^2=0$, $X_1+X_2+X_3+X_4+X_5=0$ are the five points P_i', the ten planes ϕ_{ij}, the ten lines s'_{ij}, the fifteen lines d'_{ijkl}, the fifteen planes Δ'_{ijkl}, the ten points D'_{ij}, respectively.

*Ciani, *Alcuni applicazioni geometriche della teoria dei gruppi di sostituzioni.* Giornale di Matematiche, Vol. 68, pp. 45-58.

II. The Surface of Order n Invariant Under the G_{120}

1. *Sets of Invariant Points on F_n*

A surface F_n in ordinary space which is invariant in the G_{120} is symmetric in X_1, X_2, X_3, X_4, X_5 and can be written

$$F_n \equiv \sum_{1}^{m} \lambda_i \phi_2^{r_i} \phi_3^{s_i} \phi_4^{t_i} \phi_5^{v_i} = 0$$

where $2r_i + 3s_i + 4t_i + 5v_i = n$, and ϕ_i is an elementary symmetric function of X_1, X_2, X_3, X_4, X_5 of degree i.

If we put on the condition that the sets of points invariant in the G_{120} be on F_n we find

Theorem 1. *If the 5 points P'_i or the 10 points D'_{ij} are on F_n they are at least double points.*

Theorem 2. *If n is odd, the 10 points v_{ij} are on F_n and their multiplicity is of odd order. If n is even their multiplicity is either zero, or even.*

Theorem 3. *If n is odd the 15 diagonal points are on F_n. If n is even these points either do not lie on F_n, or they are on it as double points at least.*

Theorem 4. *If n is not a multiple of 3 the 20 points $(1,\omega,\omega^2,0,0)$ etc. are on F_n. If n is a multiple of 3 these points either do not lie on F_n, or they are on F_n with a multiplicity of order 2 at least.*

(ω is a primitive cube root of unity.)

Theorem 5. *If n is not a multiple of 4 the 30 points $(1,-1,i,-i,0)$ etc. are on F_n. If n is a multiple of 4 the 30 points either do not lie on F_n, or they are on F_n with a multiplicity of order 2 at least.*

Theorem 6. *If n is not a multiple of 5 the 24 points $(1,\bar{\omega},\bar{\omega}^2,\bar{\omega}^3,\bar{\omega}^4)$, etc. are on F_n. If $n=5r+1$ the 24 points are double points, at least, of F_n. If n is a multiple of 5 the 24 points either do not lie on F_n, or they are on F_n with a multiplicity of order 2 at least.*

($\bar{\omega}$ is a primitive 5th root of unity.)

Theorem 7. *A surface invariant under the G_{120} must be of order ten or higher to have a generic point as a double point.*

The configuration of these 120 points has been studied by Veronese.* The smallest number of points invariant as a set under the G_{120} is 5. Then, if F_n has one n-fold point, or an $(n-1)$ fold it will have 5.

*Veronese, *Interprétations géométriques de la théorie des substitutions de n lettres,* etc. Annali di Matematica, series 2, Vol. 2, p. 93.

By using Hollcroft's* theorems on limits for multiple points of surfaces we can state

Theorem 8. *There are no cones of any order invariant under the G_{120} and the only moniod is $\phi_2=0$.*

2. *Sets of Invariant lines on F_n.*

The 15 diagonal lines are on $\phi_3=0$ and on $\phi_5=0$. If n is odd F_n has either a ϕ_3 or a ϕ_5 in each term. Hence

Theorem 9. *If n is odd the 15 diagonal lines are on F_n.*

The 10 edges s_{ij} and the 20 joins of the type $(1,-1,0,0,0)$ and $(0,0,1,\omega,\omega^2)$ are on $\phi_4=0$ and on $\phi_5=0$. The 12 joins of the type $(1,\bar\omega,\bar\omega^2,\bar\omega^3,\bar\omega^4)$ *and* $(1,\bar\omega^4,\bar\omega^3,\bar\omega^2,\bar\omega)$ *are on* $\phi_3=0$ *and* $\phi_2^2-5\phi_4=0$. *Hence*

Theorem 10. *The members of the pencils $\lambda_1A_i\phi_4+\lambda_2P_k\phi_5=0$, of all orders n greater than 6, except when $n=5r+4$ or $4r+5$, have the 10 lines s_{ij} and the 20 joins of the type $(1,-1,0,0,0)$ and $(0,0,1,\omega,\omega^2)$. The members of the pencils $\lambda_1c_i\phi_3+\lambda_2\phi_k(\phi_2^2-5\phi_4)=0$ of all orders n greater than 5, except when $n=4r+3$, have the 12 joins of the type $(1,\bar\omega,\bar\omega^2,\bar\omega^3,\bar\omega^4)$ and $(1,\bar\omega^4,\bar\omega^3,\bar\omega^2,\bar\omega)$. In the above A_i, B_k, C_i and D_k are functions of the ϕ's).*

The collineations of the G_{120} carry a generic line in $X_i-X_j=0$ into 60 lines which lie by sixes in the ten planes $X_i-X_j=0$. The 6 lines in $X_1-X_2=0$ intersect the join $\overline{V_{45}\ P_5'}$ which is point-wise invariant in the two perspectives (13) and (23). By the perspectives (13) and (23) each of the 6 lines in $X_1-X_2=0$ is carried into two more lines. The 18 lines then pass by threes through 6 points on $\overline{V_{45}\ P_6'}$ which are multiple points of a surface that contains the 18 lines. Since there are 10 lines $\overline{V_{ij}\ P_j'}$ a surface that contains the 60 lines will have 60 multiple points which lie by sixes on the 10 lines $\overline{V_{ij}\ P_j'}$. If the surface is of order less than 12 it will also contain the 10 lines $\overline{V_{ij}\ P_j'}$. But there are three of the lines $\overline{V_{ij}\ P_j'}$ in each plane $X_i-X_j=0$ and six of the other set. Hence

Theorem 11. *A surface, invariant under the G_{120}, which has the 60 lines into which a generic line in $X_i-X_j=0$ is carried by the collineations of the G_{120}, is of order 9 or higher and has 60 multiple points.*

The 60 lines obtained by performing the collineations of the G_{120} on a generic line through v_{ij} pass by sixes through the 10 points v_{ij}. The 18 lines which pass by sixes through the three points v_{ij} on an edge s_{kl} form 6 co-planar triples. Hence we can state

*Hollcroft, *Limits for multiple points and curves of surfaces.* Tohoku Mathematical Journal, Vol. 30, p. 115.

Theorem 12. *A surface F_n which contains the 60 lines obtained by performing the collineations of the G_{120} on a generic line through v_{ij} has 10 multiple points at the 10 points v_{ij} and sixty tritangent planes.*

Let L_1 be a generic line. The collineations of the G_{120} will carry it into 120 lines. The perspective (ij) will carry L_1 into a line L_{ij} which intersects L_1 in the points where L_1 cuts the plane $X_i - X_j = 0$. Since there are 10 perspectives (ij), the line L_1 will be intersected by 10 lines. It is easily seen that a surface must be of order 18 to have this set of lines on it. Hence

Theorem 13. *The 120 lines of the G_{120} are divided into two sets of 60 non-intersecting lines. Each line of one set is intersected by 10 lines of the other set. A surface which contains these 120 lines is of order 18 or higher.*

III. Curves Invariant Under the G_{120}

Ciani has shown that no plane curve, and no space curve of order less than 6, can be invariant in the G_{120}.* Suppose we have a space curve C_n which is invariant under the G_{120}. The C_n is invariant in the 10 involutorial perspectives (ij). Any line through v_{ij} that cuts C_n in a point P will also cut it in a point P'. If we project the C_n from v_{ij} onto a plane its points will double up giving a curve of order $n/2$. Hence the C_n lies on a cone of order $n/2$ with a vertex at v_{ij}, provided the C_n does not go through v_{ij}. If C_n goes through v_{ij}, r times the cone is of order $n-r/2$.

The n lines, joining v_{ij} to the n intersections, T_k, of C_n and $X_i - X_j = 0$, are tangent to C_n at T_k, provided the points are not multiple points of C_n. The $3n$ tangents $\overline{v_{ij}\ t_k}$ from the three points v_{ij} on an edge s_{kl} form n co-planar triples which determine n tritangent planes. If there are no multiple points on the trace of the cone (with vertex at v_{ij}) in the plane $X_i - X_j = 0$, $n-r/2(n-r/2-1)-2$ tangents can be drawn from T_k to the trace. Each of the tangents along with $\overline{v_{ij}\ T_k}$ determines a tritangent plane. Three of the tritangent planes on $\overline{v_{ij}\ T_k}$ were included in the 10 n above. Hence the C_n has $10n + [n-r/2(n-r/2-1)-2-3]10n$ tritangent planes. If there are multiple points on the trace of the cone in $X_i - X_j = 0$ the number of tritangent planes will be reduced by a number which depends on the order and type of the multiple points.

*Ciani, *La configurazione del pentædro.* Rendic. Circ. Matematico di Palermo, t. 21, p. 322.

The C_n is also invariant in the skew involution (ij) (kl). The lines joining the points of C_n to the skew lines d_{ijkl} and d'_{ijkl} are rulings of a ruled surface. On each ruling there are two points of C_n that are corresponding points of the involution. Hence the order of the ruled surface is n. If C_n cuts d_{ijkl} in s, r_1-fold points and d'_{ijkl} in s', r'-fold points the order is $n-r_1s-s'r_1'$. Hence

Theorem 14. *If C_n is invariant under the G_{120} and passes r times through the points v_{ij} it lies on 10 cones of order $n-r/2$. If C_n intersects d_{ijkl} in s,r_1-fold points it lies on 15 ruled surfaces of order $n-s'r_1-s'r_1'$. If C_n does not have any multiple points in $X_i-X_j=0$ and if the trace of the cones in the planes $X_i-X_j=0$ does not have multiple points, C_n has $10n[n-r/2(n-r/2-1)-2-3]+10n$ tritangent planes.*

IV. Quintic Surfaces Invariant Under the G_{120}

1. *The General Quintic $F_5\equiv\lambda_1\phi_2\phi_3+\lambda_2\phi_5=0$.*

If the 5 points P_i', the 10 points D'_{ij}, the 24 points $(1,\bar\omega,\bar\omega^2,\bar\omega^3,\bar\omega^4)$, etc., are on F_5 they are at least double points. The 10 points v_{ij}, the 30 points $(1,-1,i,-i,0)$, etc., the 15 points D_{ijkl}, and the 20 points $(1,\omega,\omega^2,0,0)$, etc. are on F_5. The 15 diagonal lines are on F_5. Each plane $X_i+X_j=0$ is tangent to F_5 at v_{ij}, and intersects F_5 in the three diagonal lines in that plane and a conic which degenerates when $\lambda_1=\lambda_2$. Each plane is tangent to F_5 at 9 points and intersects F_5 in three diagonal lines and a non-degenerate conic. There are 4 points of tangency on each diagonal line. The 20 planes $X_1+\omega X_2+\omega^2X_3=0$, etc., are tangent to F_5 at the 20 points $(1,\omega,\omega^2,0,0)$, etc. respectively.

2. *Some Particular Quintics of the Pencil that have Double Points.*

A proper quintic surface cannot have more than 34 double points.* The smallest number of points invariant as a set in the G_{120} is 5.

The surface $\phi_2\phi_3+50\phi_5=0$ has 5 double points P_i' which lie by twos on 10 lines s'_{ij} which pass by ones through the 10 points v_{ij} and intersect by twos the 15 pairs of skew lines d_{ijkl} and d'_{ijkl}.

The surface $12\phi_2\phi_3+25\phi_5=0$ has 10 double points which lie by twos on 30 lines which pass by threes through the 10 points V_{ij} and by twos on 45 lines which intersect by fours the 15 pairs of skew lines d_{ijkl} and d'_{ijkl}.

The surface $\phi_2\phi_3-2\phi_5=0$ has 15 double points D_{ijkl} which lie by twos on 45 lines which pass by sixes through the 10 points v_{ij} and by

*Hill, *On Quintic Surfaces.* Mathematical Review, Vol. I, pp. 1896-97.

twos on 90 lines which intersect the 15 pairs of skew lines d_{ijkl} and d'_{ijkl} by sixes.

By considering all sets of points invariant in the G_{120} we find

Theorem 15. *A quintic surface invariant under the* G_{120} *that has double points will have five, ten or fifteen.*

A quintic surface cannot have more than 5 triple points and not that many of any higher order.* The only possibility of multiple points of higher order than two is for the 5 points P'_i to be triple points of F_5. It is easy to see these points cannot be triple points of of F_5. Hence.

Theorem 16. *A quintic surface invariant under the* G_{120} *cannot have a singular point of order higher than 2.*

3. *Some Particular Quintics of the Pencil that have Lines on Them.*

A generic line cannot be on F_5 so the line must pass through two of the 120 points resulting from the G_{120}, or pass through an invariant element. We first consider the 6 types of substitutions of the G_{120}. None of the joins of the corresponding points in the 6 types of substitutions can be on a quintic surface.

We next consider the sets of lines obtained by finding the lines invariant in the subgroups of the G_{120}. The 15 diagonal lines are on all members of the pencil $\lambda_1\phi_2\phi_3+\lambda_2\phi_5=0$.

The surface $\phi_2\phi_3-\phi_5\equiv\sum_1^5 x_i^5=0$ has 35 lines, 15 diagonal lines and the 20 pairs of the type $(1,1,0,0,0)$ joined to $(0,0,1,\omega,\omega^2)$ which form 10 sets of 5 co-planar concurrent lines. The planes are the 10 planes δ_{ij} and the points of concurrence are the 10 points v_{ij}. Each of the 35 lines is intersected by 10 of the 35 lines.

Summing up this section we can state the principal results in

Theorem 17. *The sets of lines that can be on the quintic surface invariant under the* G_{120} *are the 15 diagonal lines and the 20 joins of the type* $(1,-1,0,0,0)$ *and* $(0,0,1,\omega,\omega^2)$.

V. *Sextic Surfaces Invariant Under the* G_{120}

1. *The General Sextic* $F_6\equiv\lambda_1\phi_2^3+\lambda_2\phi_3^2+\lambda_3\phi_2\phi_4=0$.

The 24 points $(1,\bar\omega,\bar\omega^2,\bar\omega^3,\bar\omega^4)$, etc. are double points of F_6. If the 5 points P_i', the 10 points D'_{ij}, the ten points v_{ij}, the 15 diagonal points,

*Hollcroft, *Limits for Multiple Points and Curves and Surfaces.* Tohoku Math. Journal, Vol. 30, p. 115.

the 20 points $(1,\omega,\omega^2,0)$, etc. are on F_6 they are double points. The 30 points $(1,-1,i-i,0)$, etc., are on F_6.

The members of the net of sextics are tangent to each other along the C_6 that is the intersection of $\phi_2=0$ and $\phi_3=0$.

These twenty-four points lie by twos on 120 lines which pass by twelves through the 10 points v_{ij} and by twos on 180 lines which intersect by twelves the 15 pairs of skew lines d_{ijkl} and d'_{ijkl}. The 24 points lie by sixes on 40 planes. The planes are $\bar{\omega}X_i-X_j=0$, $\bar{\omega}^3-X_i-X_j=0$, $\bar{\omega}^2X_i-X_j=0$, $\bar{\omega}^4X_i-X_j=0$, $i, j=1,2,3,4,5$.

2. *Some Particular Sextics of the Net that have Double Points.*

The surface $\lambda_2(8\phi_2^3+20\phi_3^2)+\lambda_3(3\phi_2^3+20\phi_2\phi_4)=0$ has 29 double points, the 5 points P_i' and the 24 above, which lie by twos on 130 lines which pass by thirteens through the 10 points v_{ij} and by twos on 190 lines which intersect the 15 pairs of skew lines d_{ijkl} and d'_{ijkl} by fourteens.

The surface $\lambda_2\phi_3^2+\lambda_3\phi_2\phi_4=0$ has 34 double points (the point s_{ij} and the 24 above) which lie by threes on 130 lines which pass through the 10 points v_{ij} and by tens through the 24 points $(1,\bar{\omega},\bar{\omega}^2,\bar{\omega}^3,\bar{\omega}^4)$, etc.

The surface $\lambda_2(4\phi_2^3+135\phi_3^2)+\lambda_3(135\phi_2\phi_4-36\phi_2^3)=0$ has 34 double points (the 10 points D_{ij} and the 24 above) which lie by twos on 150 lines which pass by fifteens through the 10 points v_{ij} and by twos on 225 lines which intersect the 15 pairs of skew lines d_{ijkl} and d'_{ijkl} by sixteens.

The surface $\lambda_3(-\phi_2^3+4\phi_2\phi_4)+4(\lambda_2)\phi_3^2=0$ has 39 double points (the 15 D_{ijkl} and the 24 above) which lie by twos on 165 lines which pass by eighteens through the 10 points v_{ij}, and by twos on 270 lines which intersect the 15 pairs of skew lines d_{ijkl} and d'_{ijkl} by eighteens.

The surface $12\phi_2^3+45\phi_3^2-4\phi_2\phi_4=0$ has 39 double points (the 5 points P_i', the 10 D'_{ij} and the 24 above) which lie by threes on the 10 lines s'_{ij} and by threes on the 15 lines d'_{ijkl}.

The surface $3\phi_3^2-8\phi_2\phi_4=0$ has 39 double points (the 5 P_i', the 10 v_{ij}, and the 24) which lie by threes on 140 lines which pass by sixteens through the 10 points v_{ij}.

The surface $\phi_2^3+4\phi_3^2-4\phi_2\phi_4=0$ has double points at the 5 points P_i' and at the 15 points D_{ijkl}. The 44 double points lie by twos on 175 lines which pass by nineteens through the 10 points v_{ij} and by twos on 280 lines which intersect the 15 pairs of skew lines d_{ijkl} and d'_{ijkl} by twenties.

The surface $9\phi_3^2+\phi_2\phi_4=0$ has double points at the 10 points v_{ij} and at the 10 points D'_{ij}. The 44 double points of this surface lie by threes on 160 lines which pass by eighteens through the 10 points v_{ij}

and by twos on 205 lines which intersect the 15 pairs of skew lines d_{ijkl} and d'_{ijkl} by sixteens.

The surface $7\phi_2^2+27\phi_3^2-12\phi_2\phi_4=0$ has 44 double points [the 20 points $(1,1,1,0-3)$ etc., and the 24 above] which lie by twos on 190 lines which pass by nineteens through the 10 points v_{ij} and by twos on 250 lines which intersect the 15 pairs of skew lines d_{ijkl} and d'_{ijkl} by twenty-twos.

The surface $4\phi_2^3-9\phi_3^2-16\phi_2\phi_4=0$ has 49 double points (10 points D'_{ij}, 15 points D_{ijkl} and the 24 above) which lie by twos on 195 lines which pass by twenty-ones through the 10 points v_{ij} and by twos on 315 lines which intersect the 15 pairs of skew lines d_{ijkl} and d'_{ijkl} by twenty-twos. There are 60 lines common to the two sets.

The surface $4\phi_2^3+27\phi_3^2-36\phi_2\phi_4=0$ has 54 double points [the 30 points $(1,1,0,0,-2)$ and the above 24] which lie by twos on 210 lines which pass by twenty-ones through the 10 points v_{ij} and by twos on 355 lines which intersect the 15 pairs of skew lines d_{ijkl} and d'_{ijkl} by twenty-sixes. The 54 double points also lie by nines in the 40 planes $\bar{\omega}X_i-X_j=0$, $\bar{\omega}^2X_i-X_j=0$, $\bar{\omega}^3X_i-X_j=0$, $\bar{\omega}^4X_i-X_j=0$ where $i,j=1,2,3,4,5$.

The 30 double points $(1,1,0,0,-2)$, etc., lie by twelves in the 5 planes π_i. Hence the planes π_i each cut the sextic surface in a plane sextic that degenerates. Hence we have

Theorem 18. *If the 30 points $(1,1,0,0,-2)$, etc., are on a sextic surface (invariant under the G_{120}) as double points each of the 5 planes π_i cuts the sextic surface in 3 conics.*

The condition that the 30 points $(1,-1,i,-i,0)$, etc., be on F_6 as double points is $\lambda_3=0$. Hence

Theorem 19. *If the condition is put on F_6 that the 30 points $(1,-1,i,-i,0)$, etc., be double points there is obtained a pencil of surfaces each of which has the intersection of $\phi_2=0$ and $\phi_3=0$ as a cuspidal curve and each member of the pencil has the same cuspidal tangents. Furthermore, the 15 planes $X_1+X_2-X_3-X_4=0$, etc., are double cuspidal tangent planes.*

The surface $2\phi_2^3+5\phi_3^2=0$ has the intersection of $\phi_2=0$ and $\phi_3=0$ as a cuspidal curve, and the 30 joins of the type $(1,1,1,1,-4)$ and $(1,-1,i,-i,0)$ as single lines.

The surface $4\phi_2^3+135\phi_3^2=0$ has C_6 as a double curve and the 10 points D'_{ij} as double points. Each plane ϕ'_{ij} and each plane Δ'_{ijk} cuts the surface in rational plane sextics.

There are no other sets of points with which the 30 points can be combined to give other proper sextic surfaces. Hence

Theorem 20. *A sextic surface invariant under the G_{120} that has double points will have 24, 29, 34, 39, 44, 49, 54, double points, a double sextic, a double sextic and 5 double points, or a double sextic and 10 double points.*

A non-degenerate sextic cannot have more than 9 triple points and cannot have as many as 5 of any higher order.* Then a sextic surface cannot have a multiple point of order higher than 3. The only set of points that could be triple points of the sextic is the 5 points P_i'. It can then be shown that

Theorem 21. *A sextic surface invariant under the G_{120}, cannot have a multiple point of order higher than two.*

3. *Some Particular Sextics of the Net that have Lines on Them.*

The members of this pencil $\lambda_1\phi_2(\phi_2^2-5\phi_4)+\lambda_2\phi_3^2=0$ are tangent to each other along a C_6 and along the 12 lines on the surfaces. The surface $2\phi_2^3+5\phi_3^2=0$ has the 30 joins of the type: $(1,1,1,1-4)$ and $(1,-1,i,-i,0)$. These lines pass by sixes through the 5 points P'_i, and each line of the set is intersected by 9 others.

The 60 lines obtained by performing the collineations of the G_{120} on the join of $(1,\bar\omega,\bar\omega^2,\bar\omega^3,\bar\omega^4)$ and $(1,\bar\omega^4,\bar\omega,\bar\omega^2,\bar\omega^3)$ lie on the surface $(\bar\omega^3+\bar\omega^2)\phi_3^2+(\bar\omega^4+\bar\omega+2)\phi_2\phi_4=0$. These 60 lines pass by sixes through the 10 points v_{ij}, and by fives through the 24 points $(1,\bar\omega,\bar\omega^2,\bar\omega^3,\bar\omega^4)$, etc. The 34 points are double points of the surface. The 60 lines lie by sixes in the 20 planes $\bar\omega X_i-X_j=0$, $\bar\omega^4X_i-X_j=0$, $i,j=1,2,3,4,5$, and by threes in the 20 planes $\bar\omega^2X_i-X_j=0$, $\bar\omega^3X_i-X_j=0$, $i,j=1,2,3,4,5$.

The other set of 60 lines is obtained by performing the collineations of the G_{120} on the join of $(1,\bar\omega^2,\bar\omega,\bar\omega^3,\bar\omega^4)$ and $(1,\bar\omega^3,\bar\omega^4,\bar\omega,\bar\omega^2)$.

The surface is $(\bar\omega^4+\bar\omega)\phi_3^2+(\bar\omega^3+\bar\omega^2+2)\phi_2\phi_4=0$. The configuration on this surface is similar to the one above. In this case the second set of 20 planes has 6 lines while the first set has 3 lines.

The pairs of points on the 60 lines of the second set can be found by performing this transformation on the pairs of points in the first set.

$$PX'_i=X_i^2,\quad i=1,2,3,4,5.$$

Theorem 22. *If a sextic surface has the join of $(1,\bar\omega,\bar\omega^2,\bar\omega^3,\bar\omega^4)$ and $(1,\bar\omega^4,\bar\omega,\bar\omega^2,\bar\omega^3)$ or the join of $(1,\bar\omega^2,\bar\omega,\bar\omega^3,\bar\omega^4)$ and $(1,\bar\omega^3,\bar\omega^4,\bar\omega,\bar\omega^2)$ it will have 60 lines, 34 double points and there will be 20 sextuply tangent planes, 20 planes which contain 6 lines and 20 contain 3 lines and a cubic.*

*Hollcroft, *Limits for Multiple Points and Curves of Surfaces.* Tohoku Math. Journal Vol. 30, p. 115.

Summing up this section we can state

Theorem 23. *The 4 sets of lines that can be on the sextic surface invariant under the* G_{120} *are: the 12 joins of the type of* $(1,\bar{\omega},\bar{\omega}^2,\bar{\omega}^3,\bar{\omega}^4)$ *and* $(1,\bar{\omega}^4,\bar{\omega}^3,\bar{\omega}^2,\bar{\omega})$, *the 30 joins of the type of* $(1,1,1,1,-4)$ *and* $(1,-1,i,-i,0)$, *the 60 joins of the type* $(1,\bar{\omega},\bar{\omega}^2,\bar{\omega}^3,\bar{\omega}^4)$ *and* $(1,\bar{\omega}^4,\bar{\omega},\bar{\omega}^2,\bar{\omega}^3)$ *the 60 joins of the type* $(1,\bar{\omega}^2,\bar{\omega},\bar{\omega}^3,\bar{\omega}^4)$ *and* $(1,\bar{\omega}^3,\bar{\omega}^4,\bar{\omega},\bar{\omega}^2)$.

At the Berkeley meetings of the Society for the Promotion of Engineering Education, held June 24-28, 1940, their Council passed the following resolution:

In various parts of the country there seems to be a movement to postpone and to abbreviate the courses in mathematics given in the secondary schools. This movement apparently does not recognize the fact that those courses are essential prerequisites for the future training of scientific and engineering students, and that the university has not postponed and cannot postpone the mathematical or the scientific and engineering instruction in the university, if its graduates are to enter those professional fields. Moreover, at the present time, for our own defense as a nation, it is suicidal not to develop the most thorough kind of training for engineers.

The members of the Conference on Mathematics of the Society for the Promotion of Engineering Education wish to go on record as recommending that there be no postponement in the mathematical education in the secondary schools of those students who are to seek careers in science and engineering. In particular, they feel that it is essential that a full four-year program of mathematics be available in the high schools for capable students, beginning with a year of college preparatory algebra in the ninth grade. They feel that this subject should not be postponed and also that thorough work in trigonometry and solid geometry should be available.

This resolution in no way implies that university preparatory courses be required of all students. But this organization feels very strongly the importance of providing substantial courses in mathematics for those who need them in preparation for future work or for those who choose to elect them. We believe that to be effective these courses must begin with algebra at least as early as the ninth year, that is, the first of the last four years in the secondary schools.—From October, 1940, issue of The American Mathematical Monthly.

The analytics test contained one question requesting the equation of the locus of a point which moves in such a way that the slopes of the lines joining it to two given points are in the ratio 3:1. On handing in his paper one bewildered freshman asked, "What is the formula for the equation of a locus?"

Evaluation of Infinite Integrals by Heaviside Operators

By J. F. Thomson
Tulane University

In a recent paper by Johnson* some features of Heaviside's operational methods are indicated. The formal solution of ordinary differential equations with constant coefficients is almost mechanical if tables of operators are used. No attempt is made in this paper to present a rigorous discussion of the convergence of series or integrals (Laplace and Fourier transforms and their inverses) involved in establishing these tables. Likewise the mathematical justification of their use is not attempted here. The book of McLachlan† (applied mathematics) and the article of Bourgin and Duffin‡ (modern research) show the present state of the theory and give many references.

It is the intent of this paper to give examples of how the table of operations may be utilized to evaluate certain infinite integrals. Heaviside's method is illustrated, a short table of operators is given and applicators are made to infinite integrals.

Heaviside's procedure is shown by the solution of the simple electric circuit problem

$$RI+L\frac{dI}{dt}=EH(t) \tag{1}$$

where R = resistance (ohms)

L = inductance (henries)

I = current (amperes)

E = applied voltage

t = time (seconds)

*William C. Johnson, Jr., *Introduction to Heaviside's Calculus*, National Mathematics Magazine, Vol. XII, 1938, pp. 231.

†N. W. McLachlan, *Complex Variable and Operational Calculus with Technical Applications*, Cambridge University Press, 1939.

‡D. G. Bourgin and R. J. Duffin, *The Heaviside Operational Calculus*, American Journal of Mathematics, Vol. 59, 1937, pp. 489-505.

R, L, E constants. $H(t)$ is the Heaviside unit function defined as $H(t)=0$, $t<0$; $H(t)=1$, $t>0$. We write p in place of d/dt and define $1/p$ by

$$\frac{1}{p}\varphi(t)=\int_0^t \varphi(t)dt$$

For $t>0$
$$\frac{1}{p}H(t)=\int_0^t dt=t$$

$$\frac{1}{p^2}H(t)=\frac{1}{p}\cdot\frac{1}{p}H(t)=\int_0^t tdt=\frac{t^2}{2}$$

$$\cdots\cdots$$

(2)
$$\frac{1}{p^n}H(t)=\frac{t^n}{n!}$$

(1) may be rewritten

$$I=\frac{1}{R+pL}EH(t)=\frac{E}{L}\quad\frac{1}{p+\alpha}H(t),\text{ where }\alpha=\frac{R}{L}.\quad\frac{1}{p+\alpha}$$

is called the *impedance operator* by electrical engineers. In more general problems $p+\alpha$ is replaced by a polynomial $Z(p)$; the corresponding operator

$$\frac{1}{Z(p)}$$

acting on $H(t)$ gives the Heaviside solution $I=A(t)$ of

$$Z(p)I=H(t).$$

The solution of (1) with the initial condition $I=0$ when $t=0$ is obtained from

$$\frac{1}{p+\alpha}H(t)=\left[\frac{1}{p}-\frac{\alpha}{p^2}+\frac{\alpha^2}{p^3}-\cdots+(-1)^{n-1}\frac{\alpha^{n-1}}{p^n}+\cdots\right]H(t)$$

$$=t-\frac{\alpha t^2}{2!}+\frac{\alpha^2t^3}{3!}-\cdots+\frac{(-1)^{n-1}t^n}{n!}+\cdots$$

$$=\frac{1}{\alpha}\left[\alpha t-\frac{(\alpha t)^2}{2!}+\cdots+(-1)^{n-1}\frac{(\alpha t)^n}{n!}+\cdots\right]$$

$$=\frac{1}{\alpha}(1-e^{-\alpha t})$$

$$I=\frac{E}{R}(1-e^{-(Rt/2)})$$

Lists of such operators may be found in books on electrical circuit theory.*,†,‡.

Five such operators are listed below: $(t>0)$

$$\text{(I).}\quad \frac{1}{p^n}H(t)=\frac{t^n}{n!}$$

$$\text{(II).}\quad \frac{1}{p+\alpha}H(t)=\frac{1}{\alpha}\,[1-e^{-\alpha t}]$$

$$\text{(III).}\quad p^{-\frac{1}{2}}H(t)=2\sqrt{\frac{t}{\pi}}$$

$$\text{(IV).}\quad \frac{p^2}{p^2+w^2}H(t)=\cos wt$$

$$\text{(V).}\quad p^{\frac{1}{2}}H(t)=\frac{1}{\sqrt{\pi t}}\quad .$$

The inverse of

$$\frac{1}{Z(p)}\,H(t)=A(t) \tag{3}$$

is furnished by the infinite integral theorem of Heaviside

$$\frac{1}{aZ(a)}=\int_0^\infty e^{-ax}A(x)dx,\quad a>0 \tag{4}$$

A discussion of (4) may be found in the books on circuit theory previously mentioned.

*E. J. Berg, *Heaviside's Operational Calculus*, McGraw-Hill, 1929, p. 164.
†Vannevar Bush, *Operational Circuit Analysis*, John Wiley, 1929, p. 380.
‡J. R. Carson, *Electric Circuit Theory*, McGraw-Hill, 1926.

If we rewrite

(I). $\frac{n!}{p^n} H(t) = t^n,$ we obtain $\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$

(III). $\frac{\sqrt{\pi}}{2p^{1/2}} H(t) = \sqrt{t},$ $\int_0^\infty e^{-ax}\sqrt{x}\, dx = \frac{\sqrt{\pi}}{2a^{3/2}}$

(IV). $\frac{p^2}{p^2+m^2} H(t) = \cos mt,$ $\int_0^\infty e^{-ax} \cos mx dx = \frac{a}{a^2+m^2}$

(V). $\sqrt{\pi}\, p^{1/2} H(t) = \frac{1}{\sqrt{t}}$ $\int_0^\infty \frac{e^{-ay}}{2\sqrt{ay}} dy = \frac{\sqrt{\pi}}{2a}.$

The probability integral $\int_0^\infty e^{-a^2x^2} dx$

goes over under the change of variable $y = ax^2$ into

$$\int_0^\infty \frac{e^{-ay}}{2\sqrt{ay}}\, dy$$

which is evaluated by means of (V).

It is hoped that this paper will help to stimulate an interest in this field of applied mathematics.

Every teacher has experienced in his own classes the counterpart of one of the dimmer lights in a certain trigonometry class, who asked what was, to him, clearly a question of considerable import, "Is it true that there are fifty-nine formulas in trigonometry?"

Humanism and History of Mathematics

Edited by
G. Waldo Dunnington

A History of American Mathematical Journals

By Benjamin F. Finkel
Drury College

(Continued from December, 1940, issue)

THE MATHEMATICAL MISCELLANY

Following the *Mathematical Diary*, the next mathematical periodical to appear in the United States was the *Mathematical Miscellany*, edited and published by C. Gill, Professor of Mathematics in the Institute at Flushing, Long Island.

The *Miscellany* was a semi-annual publication. The writer of this history was able to acquire Volume I and the two numbers of Volume II, eight numbers in all,—apparently, all that were ever published, from Dr. Artemas Martin, Washington, D. C. In attending the Bicentennial Conference of the University of Pennsylvania, in Philadelphia, September 16-20, 1940, the writer took occasion to re-examine the copy in the Library of the University and found this copy to consist of only eight numbers.

Bolton reports that there are copies in the following libraries: Amherst College, Yale University, Hamilton College, Columbia University, New York Public Library, Library of the U. S. Coast and Geodetic Survey Office, Library of the U. S. Naval Observatory, and Riggs Memorial Library of Georgetown University.

Following the title page is the Table of Contents of the first volume. The two volumes of the *Miscellany* are divided into two departments, a Junior Department and a Senior Department. The departments are divided into Articles. The Junior Department of Volume I contains fourteen articles and the Senior Department, thirty.

The following is a copy of the title page:

THE
MATHEMATICAL
MISCELLANY

NUMBER I.

Conducted by
C. GILL,
Professor of Mathematics in the Institute at Flushing, Long Island.

Published at the Institute.

New-York:
W. E. DEAN, Printer, No. 2 Ann-Street.

1836.

Following the Table of Contents is a note in which it is stated that Mr. George R. Perkins, of Clinton, N. Y., desires a situation as teacher of mathematics. He became an active contributor to the *Miscellany* and from his address as given in Volumes I and II, he became a teacher of mathematics in the Clinton Liberal Institute and later, Professor of Mathematics, Utica Academy, N. Y.

Here is a case, perhaps, where a young man was led to continue his study and interest in mathematics through the influence of his own contributions and the study of the contributions of other contributors to the *Mathematical Miscellany.*

Such cases may be multiplied by hundreds where young men have been led to continue their study of mathematics through the influence of such mathematical magazines as the *Mathematical Miscellany.*

While the Table of Contents of Volume I and this note follow the Title Page, the second page of the sheet is numbered 414.

Following the Table of Contents of Volume I, the Editor has this to say;

Sir,

I have taken the liberty of sending you a copy of the first number of the Mathematical Miscellany, and hope the design of the work will meet with your approbation.

Should you feel inclined to contribute towards its support by becoming a subscriber, you are respectfully requested to inform me, by mail, of your intentions, before the fifteenth of June.

If sufficient patronage be obtained to justify the continuance of the work, agents will be appointed in different parts of the Union to receive the amount of the subscription, and the second number will be published on the first day of October next.

I am, Sir,

Yours very respectfully,

Institute at Flushing, L. I.,
March 24, 1836.

C. GILL.

Following the preceding page is a record of Meteorological Observations, made at the Institute, Flushing, L. I., for thirty-seven successive hours, commencing at six A. M. of the twenty-first of March, Eighteen-hundred and thirty-six, and ending at six P. M. of the following day.

(*Lat. 40°44'58" N.; Long. 73°44'20" W. Height of barometer above low water mark Flushing Bay, 54 feet.*)

Pages III to IV contain the following:

In the *Advertisement*, pages III and IV, the editor says that the *Mathematical Miscellany* is an experiment and "presented to the public without the courtesy of a prospectus, in the belief that its character and claims to patronage will be better understood by giving a specimen, than a promise of what it will be."

"The high estimation in which such works are held by the European mathematicians, and the fact that the great improvements in Analysis, and the vast variety of elegant problems scattered through the elementary works on science in present use, have generally first appeared in them, and are principally due to the discussions and investigations they are calculated to bring forward, are at least presumptive evidence that a miscellany of this kind might be made sufficiently interesting, were the talent of the country, of which there is certainly no want, concentrated in its aid."

"The advantages of such a work, as a medium for valuable communications that might otherwise be lost to the public; as an index to mark the taste in sciences and the progress in discovery, of the day and of the country; and as a field where the aspirant to mathematical distinction may try his strength with those of established reputation, will be perceived at once by all who would think of patronizing this undertaking."

"The Editor has the assurance of assistance from individuals whose names would be a sufficient guarantee for the respectability of the work; and if he succeeds in establishing it, he has no doubt of

enlisting in its aid much of the mathematical talent of the United States. He begs leave to commend his undertaking, in particular to gentlemen of the mathematical chairs in our colleges, with the suggestion, whether it might not be made a useful auxiliary in cherishing a spirit of science in their classes. Should this suggestion meet their view, there will be formed a distinct department adapted to this purpose; and pains will be taken to make this part of the work interesting, for it will be peculiarly gratifying to the Editor, if he can supply the means in any degree of fostering the emulation of American youth in a study which is peculiarly adapted to the inquiring mind, and which is directly becoming of more practical importance to the country.

"The *Mathematical Miscellany* will appear semi-annually on the first days of March and October; thus making the summer interval of seven months, and the winter one of five; a distinction which will be at once appreciated by the ***student***. The price of each number will be 50 cents, and as it is not designed to secure any profit from the publication, the size of the work will be increased to whatever extent its sales will allow."

Institute at Flushing, L. I.,
February, 1836.

Volume I of the *Mathematical Miscellany* consists of six semi-annual numbers and contains 412 pages. We shall give a fairly complete account of the contents of the eight numbers, for the reason that some of the readers of these pages, may be interested enough to look up the discussions in some of the articles some time in the future.

Each number of the volumes has at the top of the first page in large type, *The Mathematical Miscellany*. Accordingly, the numbering of the pages, Volume I, begins on page 5, with Art. I, Investigations of a formula for finding the Longitude at sea, by Δ, pages 5 to 10. Art. II, Solutions of a Geometrical Problem, by Δ, pages 10 to 17, Art. III, Illustrations of Lagrange, pages 18 to 29. It is not clear who the author of this article was. The following Problem I, page 18, is proposed: Upon a horizontal plane, a rectilineal path is traced in which a body P is constrained to move uniformly. This body is connected by an inflexible and inextensible line, with another body M, which is posited on this plane and which is supposed to have received some primitive impulse in the direction of this plane. It is required to find the nature of the curve described by the body M and the other circumstances of the motion, abstracting from friction.

The Proposer after giving a brief history of the problem gives his own solution in which this equation occurs,

$$c' \cos \alpha + c \sin \alpha = b \cos \alpha \cdots (13),$$

On page 23, *Some explanations of M. Francais' equation* (13) *of conditions*, by M. Dubuat. Then on page 25 problem II is proposed and reads as follows:

A uniform straight rod AB is placed in an assigned position, upon a smooth horizontal plane, and one end of it, B, is drawn uniformly along a straight line of the plane, with a given velocity; it is proposed to find the position of the rod at any time, and its angular velocity.

The Proposer then states that the question was proposed in the *Ladies' Diary* for 1826, by Mr. Mason, of Scoulton, a gentleman whose labors have enriched the English periodicals for several years. There are two solutions by the Proposer inserted in the *Diary*, neither of which have any resemblance to the following one. Then follows the Proposer's solutions, pages 25 to 29. It is probable that the Proposer was Δ.

Art. IV, On Spherical Geometry, by Δ, pages 29 to 52 (to be continued).

Δ's introduction to his subject has interest. He says, the singular analogy which exists between many of the results in Plane and Spherical Geometry, has been noticed by several writers on these subjects; and it has no doubt occurred to many that the system of Analysis, which has been gradually perfecting by the successors of Leibnitz, might possibly be adapted to the investigation of lines on the sphere. A short paper on this subject in the *Ladies' Diary* for 1835, by T. S. Davis, Esq., is introduced, with the following remarks:

"The fertile mind of Euler seems to have been the first that conceived the idea of spherical co-ordinates; and it occurred to him as the means of eluding a difficulty of a kind that was then peculiar, (in his first paper on the Halleyan Lines, *Berlin Mémoires* for 1756) and, with the occasion that gives rise to it, he laid it aside, and seems never more to have resumed the employment of the method, or the investigations of its principles. About the close of the last century, several mathematicians of great eminence in this country also entered upon the inquiry; but, owing to the awkwardness of the trigonometrical notation that then prevailed, they did not find the results of such a kind as to encourage them to proceed to any great extent.......

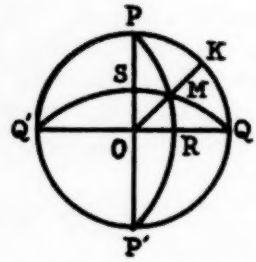
"About 1827, I was led to consider the nature of the hour lines upon the antique sun-dials. The successful application of the method to this previously intricate question, led me to investigate the principles of spherical geometry more carefully. In two papers in Volume XII of the Transactions of the Royal Society, Edinburgh, I have given the results of these researches, so far as *polar spherical co-ordinates*

are concerned, with several applications of the method as well as in one or two other places; but especially in the number now printing of the Monthly Repository, by my excellent friend Professor Leybourn."

Some idea may be formed of the light in which Mr. Davis has viewed the subject, from the following definitions, in the same article.

Conceive that O is the pole (or spherical centre) of the great circle PQP′Q′; and that M is a point on the spherical surface which we desire to refer to spherical co-ordinates.

1. If we take OR as a primitive meridian through O, the origin of co-ordinates, then an equation between the angle QOK (or arc KQ) and the polar distance OM, (spherical radius vector) will express the nature of the curve in which M may be supposed to be situated. This system is that of *spherical polar co-ordinates.*



Sometimes it is convenient to consider the relation between QK and KM, in preference to the other system.

2. We may consider the point M defined by the relation between OS and OR. This sytem is called the *longitudinal system of spherical co-ordinates,* the arcs OS and OR being measured in the same manner as longitudes on the earth are measured.

3. The point M may also be defined by an equation between MS and MR. This system, from the arcs being measured as terrestial latitudes, I have called the *latitudinal system of spherical co-ordinates.*"

He (Davis) then gives the formulas for transforming the co-ordinates from one system to either of the others, and promises to conclude the subject in a succeeding number of the *Diary.* I have endeavored, but in vain, to procure the volume of the Edinburgh Transactions to which Mr. D. alludes, and I believe the number of of the Repository (I presume he means the *Mathematical Repository*) that he refers to as being then printed, has never yet been published. I have, therefore, been induced to take up the subject myself; and presuming that other American readers will find the same difficulty in procuring access to Mr. Davis' researches as I have done, I have here endeavored to arrange the results at which I have arrived, as well as many that have laid by me for many years, into something like a regular system, and shall offer them to the readers of the *Miscellany* in this and the succeeding numbers.

I have used entirely the "*spherical polar co-ordinates*", not only because the subject first presented itself to my own view in this light, but because there still appears a greater facility of investigation by this system than by either of the others. I shall, however, take occasion to show the mode of transforming an equation from one system to another, and point out some remarkable analogies exhibited in the results." Δ.

Art. V, New Questions to be answered in Number II, 20 in all, pages 53 to 56, Note by the Editor, page 56, in which he states that all communications for the second number of the *Miscellany* must arrive before the first of August, 1836.

Then on an unnumbered page following 56, the Editor has this to say

TO SUBSCRIBERS:

On the fifteenth of June the sum arising from the subscriptions for the *Mathematical Miscellany* did not amount to one-fourth of its probable expenditure. The design of continuing it would therefore have been abandoned, had not several gentlemen from different parts of the United States, and independently of each other, proposed that the subscription price of the work should be raised, to \$1.00 per number, or \$2.00 per annum; and that any deficiency which might still exist should be sustained by those interested in the continuance of the publication, in the proportion of certain annual sums to be named by themselves individually.

The following is a list of the gentlemen who pledge themselves to support the work to the extent of the sums placed opposite their names:

Rev. W. A. Muhlenburg, D. D., Principal of the Institute, Flushing	\$50.00	per annum
C. Gill, Professor of Math., Institute, Flushing	20.00	"
L. Van Bokkelen, Math. Inst., Institute, Flushing	10.00	"
G. B. Docharty, Math. Inst., Institute, Flushing	10.00	"
Prof. C. Avery, Hamilton College, Clinton, N. Y.	15.00	"
Prof. M. Catlin, Hamilton College, Clinton, N. Y.	10.00	"
O. Root, Mathematical Tutor, Hamilton College, Clinton, N. Y.	10.00	"
G. R. Perkins, Liberal Institute, Clinton, N. Y.	5.00	"
William Lenhart, Esq., York, Penn	10.00	"
J. F. Macully, Esq., New-York	5.00	"
Evans Hollis, Esq., Westchester Co., N. Y.	5.00	"

The advanced price of subscription is certainly not more than in justice it ought to be; since the cost of printing such a work is more than double that of ordinary matter. If this advance is not objected to by the subscribers, the *Miscellany* may be considered as firmly established.

I have found it impossible to procure agents for the work in different states of the Union, the circulation being necessarily so small as to hold out no inducement to men of business. It will be therefore necessary for subscriptions to be sent by mail, and where two or more subscribers reside in a place, by enclosing the joint amount, they will not find this method inconvenient. In the case of a single subscriber, he is requested to transmit $5.00 on the receipt of the 2nd, 7th, 12th, &c., numbers.

In the peculiar circumstances under which this work is published, the necessity of having its accounts settled at stated periods will be immediately seen. In all cases, therefore, where subscriptions are not remitted by the first of January of every year, the subscription will be considered as having ceased, and the work will be no longer forwarded.

Flushing, L. I., October, 1836. C. GILL.

Following this sheet there is a sheet both sides of which contain Meteorological Observations at the Institute.

Then follows the *Mathematical Miscellany*, number II, beginning with page 57 and ending with page 128.

Page 57, Junior Department: The Editor says, this part of the *Miscellany* will be adapted to the ordinary mathematical attainments of Youth in the College Classes of our country. It will occasionally contain articles elucidating principles, and exhibiting methods, of arithmetical and analytical processes, better adapted to ordinary purposes than are generally found in our text books. In this way we shall endeavor to lend some aid to our brethren in the business of instruction, hoping to receive from them hints of the same kind in exchange, which we shall gladly publish.

The department will consist chiefly of questions calculated to interest the tyro in science, and thus perhaps be a further help to instructors in drawing out and encouraging the latent and unfolding talent which must so largely exist in our literary institutions. With proper co-operation in this object, which we earnestly solicit from the mathematical professors who patronize the *Miscellany*, we hope to supply a stimulus to industry, not afforded by the duties of the recitation room. The publication of a neat solution will come to be considered by the young aspirant to scientific distinction, as sufficient reward for the labor of preparing it.

Article I, Hints to Young Students, pages 58-62. Article II, Questions to be answered in Number III, six in all, page 62.

Senior Department: pages 63 to 128. Article VI, Solutions to questions proposed in Number I, pages 63-108.

Pages 107 and 108 contain the names of the 22 contributors to the questions in No. I, Art. V. Among them are Professor C. Avery, and Professor Catlin, Hamilton College, William Lenhart, York, Pa., George R. Perkins, Clinton Liberal Institute, N. Y., Professor B. Peirce, Harvard University, and Professor T. Strong, LL.D., Rutgers College.

Then follows a note by the Editor in which he says, In selecting from this valuable collection of solutions, we have been often obliged to exclude solutions which we should have been glad to publish, merely because they were left in an incomplete state by their author. We would respectfully remind our correspondents that solving a question and telling how it might be solved, are two very distinct things. When a question does not involve some new principle in science, and such cannot always be expected in a work like the present, its solution can only be valuable, either to the writer or reader, in so far as it affords an exercise in analysis; and such a solution must be very unsatisfactory to the reader, when the most difficult part of the analysis is left unfinished.......

Article VII, New questions to be answered in Number III, 15 in all, pages 109-111. Article VIII, New Questions to be answered in Number IV, 15 in all, pages 111-113. Art. IX, A new demonstration of the Logarithmic Theorem, and of the Binomial Theorem for negative and fractional exponents; by Professor Marcus Catlin, Hamilton College, N. Y., pages 113 and 114. Art. X, A general investigation with reference to the construction of a Table of Numbers, and the roots of two cubes of which they are composed, and practical illustrations of the equation

$$X^3+Y^3=(X+Y)(X^2-XY+Y^2)$$

when $X+Y$ is a cube, or a multiple of a cube, or nine times a multiple of a cube; and also when X^2-XY+Y^2 is a cube, or a multiple of a cube: together with the solutions of two general Problems, and their application to several Examples. By William Lenhart, Esq., York, Penn., pages 114-128.

The Mathematical Miscellany. Number III, Junior Department, pages 129-141. Art. III, Hints to Young Students (continued), pages 124-141. Art. IV, Solutions to the questions proposed in Number II, pages 136-140. Art. V, Questions to be answered in Number IV, 5 in all, page 141.

Senior Department: pages 141 to 194, Art. XI, Solutions to the questions proposed in Art. VII, pages 141-178. New Books, page 178. List of contributors page 179.

Art. XII, New questions to be answered in Number V, in all 16, pages 179-181. Art. XIII, On Spherical Geometry (continued from Art. IV, p. 52) § IV, pages 182-194. The two pages on the following sheet, Meteorological Observations at the Institute. Then a folding sheet containing a page of Geometrical Diagrams belonging to No. III.

The Mathematical Miscellany, Number IV, pages 197 to 268.

Junior Department, Art. VI. Solutions to the questions proposed in Number III, pages 197-203, Art. VII, questions to be answered in Number V, 6 in all, pages 203-204. Art. VIII, Hints to Young Students (continued from page 135), pages 204-210.

Senior Department, pages 210 to 268. Art. XIV, Solutions to the questions proposed in Art. VIII, pages 210-255. List of contributors and New Books, page 256.

Art. XV, New questions to be answered in Number VI, 15 in all, pages 257-258. Art. XVI, Note on question (35) of the *Mathematical Miscellany* by Dr. Strong, pages 259-260. Art. XVII, On Forces by Professor Harnay, Illinois College, South Hanover, Illinois, pages 261-262. Art. XVIII, Diophantine Speculations, by Wm. Lenhart, Esq., York, Penn., Number One, pages 263-267.

Page 268, Note by the Editor. "It has occurred to us that some of our correspondents may like to see the question proposed by Mr. Abbot, referred to on page 256. Mr. Abbot states that it was suggested to him by seeing the attempt made to blow two cards asunder, by a tube inserted in one of them.

EXPERIMENT: Let a circular orifice of a given diameter be made in the plane side of a deep cistern, which is kept filled with water, and a circular plate be placed over the orifice, supported at the bottom, so that its weight will not cause it to slide down. A thin sheet of water will issue between the side of the cistern and the plate, on all sides, the pressure of the atmosphere, under ordinary circumstances, causing the sheet to be unbroken between the extremity of the plate and the orifice, and the sheet to adhere to the side of the cistern, separated by the thin sheet of water.

Having given the depth of the orifice below the surface of the water, it is required to find the diameter of the smallest circular plate which can be made to adhere, and also the force with which a plate of any given diameter adheres. The plate and the sides of the cistern are supposed perfectly smooth, and the orifice and plate are concentric.

A few errata follow. Also following page 268, is a sheet the two pages of which give the two Meteorological Observations at the Institute, June twenty-first.

The Mathematical Miscellany. Number V, pages 269 to 342.

Junior Department, pages 269-284. Art. IX, Hints to Young Students (continued from page 210) pages 269-284, 21. Elevation of Powers: Extraction of Roots, pages 269-284.

Art. X, Solutions to the Questions proposed in Number IV, pages 276-384. List of contributors and a note by the Editor, page 325.

Art. XX, New Questions to be answered in Number VII, 16 in all, pages 326-328, Art. XXI, Demonstration of a Theorem in Geometry, by Dr. Strong, page 329, Art. XXII, Diophantine Speculations, by Wm. Lenhart, Esq., York, Penn., Number Two, pages 330-336, Art. XXIII, On the Theory of exponential and Imaginary Quantities by W. S. B. Woolhouse, F. R. A. S., Actuary of the National Loan Fund Life Assurance Society, London. (From the *Gentlemen's Diary,* for 1837), pages 336-342. Then follows a sheet the two pages of which contain the two Meteorological Observations at the Institute for December twenty-first, 1837.

The Mathematical Miscellany, Number VI, November 1, 1838, pages 343 to 412.

Junior Department, pages 343 to 360. Art. XII, Hints to Young Students (continued from page 276), pages 343-349, Art. XIII, Solutions to questions proposed in Number V, pages 349-359, Art. XIV, Questions to be answered in Number VII, 12 in all, pages 359-360.

Senior Department, pages 361-412, Art. XXIV, Solutions to the questions proposed in Art. XV, Number IV, pages 361-397, Art. XXV, New Questions to be answered in Number VIII, in all 15, pages 397-399, Art. XXVI, Another solution of question (51), page 399, Art. XXVII, Solution of a problem, by Professor J. H. Harney, Louisville, Ky., pages 399-400, Art. XXVIII, On the orthographic Projection of the circle, by Dr. T. Strong, New Brunswick, N. J., pages 401-403, Art. XXIX, An account of Mr. Talbot's "Researches in the Integral Calculus", published in the Philosophical Transactions, London, 1836, 1837; with a more general solution of the Principal Problem by Professor Benjamin Peirce, Harvard University, Cambridge, pages 404-411, Art. XXX, Another solution to question (50) by Dr. T. Strong, Rutger's College, New Brunswick, N. J., pages 411-412. Useful Tables relating to Cube Numbers. Calculated and Arranged by William Lenhart, York, Penn. Designed to accompany his general investigation of the equation

$$X^3+Y^3=(X+Y)(X^2-XY+Y^2)$$

published in the *Mathematical Miscellany*, Vol. I, page 114; and by him, through his friend Professor C. Gill, presented to the Library of St. Paul's College, Flushing, Long Island, May 4th, 1837. "There are few difficulties which will not yield to perseverence." New York; Printed by Wm. Osborn, 88 William Street, 1838. 18 pages including the title page of the Tables.

Following the title page of the Tables it is stated that besides the Tables given here, the manuscript copy compiled with so much labor and care, by Mr. Lenhart includes a Table, "Containing a variety of numbers between 1 and 100,000, and the roots, not exceeding two places of figures, of two cubes, to whose difference the numbers are respectively equal"; together with a table, "Exhibiting the roots of three cubes to satisfy the indeterminate equation of $X^3+Y^3+Z^3=A$, for all values of A, from 1 to 50, inclusive."

"Both these Tables are extremely curious, and are open to the inspection of all who may care to consult them. They are lodged in the Library of St. Paul's College."

Also on this page is a list of errata.

The following is the Title Page of Volume II of the *Mathematical Miscellany*.

THE

MATHEMATICAL

MISCELLANY

VOLUME II.

Conducted by

C. GILL

Professor of Mathematics, St. Paul's College, N. Y.

Published at St. Paul's College

For Sale In New-York,

by

William Jackson, 102 Broadway;
And in London;

by

Richard Groombridge, Panyer Alley, Paternoster-Row.

1839.

The
MATHEMATICAL MISCELLANY
Number VII, May 1, 1839

Junior Department, pages 3-26. Art. I, Hints to Young Students, No. VI, pages 3-12, Art. II, Solutions to questions proposed in Number VI, pages 13-23. List of Contributors, page 24. Art. III, Questions to be answered in Number VIII, 12 in all, pages 24-26. Senior Department, pages 26-74. Art. I, Solutions to the Questions proposed in Number V, pages 26-61. List of Contributors, page 60. Two notes by the Editor, page 61. Art. II, New Questions to be answered in Number IX, in all 15, pages 61-63. Art. III, Motion of a system of Bodies about a Fixed Axis, by Professor T. Strong, LL.D., New-Brunswick, N. J., pages 64-71. Also a Note by the Editor, page 71 in which he says that "The Title to the Last Article should be Motion of a System of Bodies round a fixed point. We have also been obliged to defer Mr. Macully's Article on the Summation of Trigonometrical Series. It will be inserted in the Next Number." Meteorological Observations, pages 72-74.

The Mathematical Miscellany. Number VIII, November 1, 1839, pages 73-142. Junior Department, pages 73-—, Art. IV, Solutions to the Questions proposed in Number VII, pages 81-88, Art. VI, Questions to be answered in Number IX, pages 88-90. Senior Department, Art. IV, Solutions to the Questions proposed in Number VI, pages 90-114. List of Contributors, page 115. On the Same page, the Editor says, "All communications for Number IX, which will be published on the first day of May, 1840, must be postpaid, addressed to the Editor, College Point, N. Y., and must arrive before the first of February, 1840. New Questions must be accompanied with their solutions." This last restriction is regrettable as it affords little opportunity for "Note by the Editor."

Art. V, New Questions to be answered in Number X, 15 in all, pages 115-117, Art. VI, Summation of Trigonometric Series, by J. F. Macully, Esq., New York, pages 118-126, Art. VII, Diophantine Speculations, by Wm. Lenhart, Esq., York, Pa., pages 127-132, Art. VIII, On the application to Sturm's Theorem, pages 133-149, Art. IX, Note on a Continued Product, by the Editor, pages 140-142.

Thus ends, abruptly, Vol. II of the *Miscellany* and so far as the Writer of this history was able to find, the end of the *Mathematical Miscellany* as well.

The *Mathematical Miscellany* was ably edited and conducted in a gentlemanly and courteous manner.

It thus attracted to its pages the ablest and most distinguished mathematicians of its day and these men by their contributions to the *Mathematical Miscellany* not only enriched it, but through its influence, enriched the whole domain of mathematics in the United States.

The following are some of the men who contributed most regularly to its pages:

Professor C. Avery, Hamilton College, N. Y.; P. Barton, Jr., Esperance, N. Y.; B. Birdsall, Clinton Liberal Institute, N. Y.; Professor M. Catlin, Hamilton College, Clinton, N. Y.; E. H. Delafield, St. Paul's College, N. Y.; D. Kirkwood, York, Pa.; Wm. Lenhart, York, Pa.; J. F. Macully, teacher of mathematics, New York; Professor B. Peirce, Harvard University; Professor G. R. Perkins, Utica Academy, N. Y.; Professor T. Strong, LL.D., Rutger's College, New Brunswick, N. J.

Some of the contributors were too modest to sign their proper names, so used pseudonyms.

Of these, perhaps the most versatile and prolific contributor was Professor Benjamin Peirce, Professor of Mathematics in Harvard University. There are also a great many solutions of problems in all branches of mathematics by Professor Theodore Strong, Rutger's College, New Brunswick, N. J., Professors Charles Avery and Marcus Catlin of Hamilton College, Clinton, N. Y., and Professor George R. Perkins of whom mention has already been made. William Lenhart, an acknowledged authority on the Diophantine Analysis, contributed many problems and solutions as well as original articles on that difficult subject.

The Society for the Promotion of Engineering Education was scheduled to hold its ninth annual meeting at the University of Southern California in Los Angeles on December 27-28, 1940. Professor D. B. Prentice of Rose Polytechnic Institute in Terre Haute, Indiana, is president of the Society. Professor Philip S. Biegler of the University of Southern California is chairman of the Pacific Southwest Section.

The fourth annual William Lowell Putnam Mathematical Competition, under the sponsorship of the Mathematical Association of America, will be held on Saturday, March 1, 1941. Any college or university wishing to enter a team or individual contestants may secure an application blank from Professor W. D. Cairns, 97 Elm Street, Oberlin, Ohio.

The journal *Mathematical Reviews* offers a reading machine for microfilm to any person who has paid his subscription, at the rate to which he is entitled, to *Mathematical Reviews* in advance for three years beginning January, 1941. The person who receives a reading machine must pay express charges from Buffalo, New York. The supply is limited.

—Reported by L. J. Adams.

The Teacher's Department

Edited by
JOSEPH SEIDLIN and JAMES MCGIFFERT

The Trisection Problem

By ROBERT C. YATES
Louisiana State University

CHAPTER II

SOLUTIONS BY MEANS OF CURVES

From the very beginning, keen sighted persons suspected the impossibility of a solution of the Trisection Problem through the medium of straight lines and circles and looked about for other means to turn the trick. Since these two curves, the line and circle, were found insufficient, one person after another began to devise new and more complex curves, thus of course breaking the rules of the game as laid down by Plato. Many of these curves did offer solutions to the problem and, in addition, played important roles in other fields of mathematics and physics. For these reasons, they deserve a prominent place in our discussion. The drawing of these curves called for more complicated tools than the simple straightedge and compasses and their description forms a part of the subject of the next chapter.

1. The Quadratrix

The Quadratrix, invented by Hippias in an attempt to trisect the angle and square the circle, is formed in the following fashion. In Fig. 3, COB is a quadrant of the unit circle. The point D travels along the line from O to C at a constant rate. In the same interval of time, the point E moves from B to C along the arc, also at a constant rate. The horizontal line through D meets OE in P. The path described by P is the Quadratrix.

This is the second in a series of five chapters. The third will follow in an early issue.

It is evident from this definition that the ratio of the lengths of any two arcs BE and BA is the same as the ratio of their corresponding segments on OC. That is,

$$OD/OF = BE/BA = \Theta/\phi. \tag{1.1}$$

Having by some means drawn the curve, if $AOB = \phi$ is the angle to be trisected, it is necessary only to take $OD = (1/3)OF$, in the manner of Fig. 2. Thus from (1.1):

$$(1/3)(OF/OF) = \Theta/\phi,$$

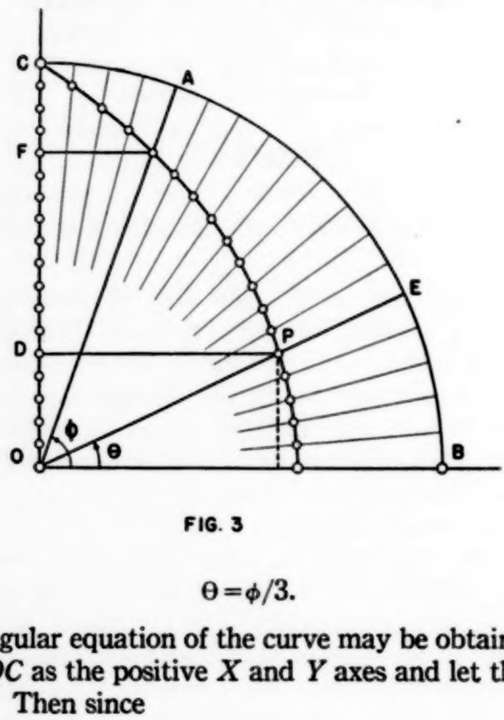


FIG. 3

or

$$\Theta = \phi/3.$$

The rectangular equation of the curve may be obtained as follows. Take OB and OC as the positive X and Y axes and let the coordinates of P be (x,y). Then since

$$OD/OC = \Theta/(\pi/2), \quad x = (OP)\cos\Theta, \quad OC = 1, \quad OD = y,$$

we have:

$$x = y\cot\Theta \quad \text{and} \quad y = 2\Theta/\pi.$$

These form the *parametric* equations of the Quadratrix and the *rectangular* equation results from eliminating Θ:

$$\boxed{y = x\cdot\tan(\pi y/2)}\,.$$

The reader familiar with indeterminate forms will find that the curve strikes the line OB at a point $2/\pi$ units distant from O.

2. *The Conchoid*

The Conchoid, designed by Nicomedes about 200 B. C., was used to obtain a solution of the Trisection Problem by Pappus five centuries later. It is formed in a very simple way. A circle moves with its center always on a fixed line XX. Through its center and also through a fixed point O, not on the line, passes the line OT. The path of the intersections P and P' of this line with the circle is the Conchoid. There are thus two branches of the curve, both having the line XX as an asymptote.

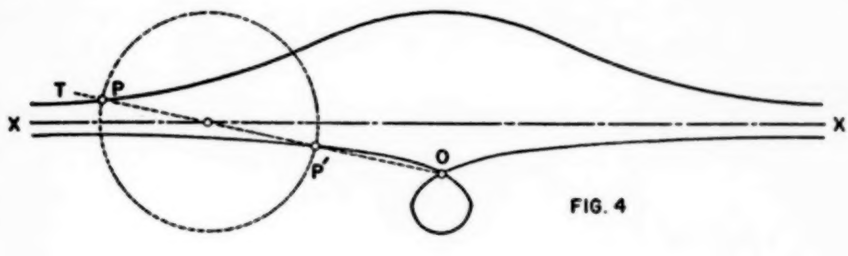


FIG. 4

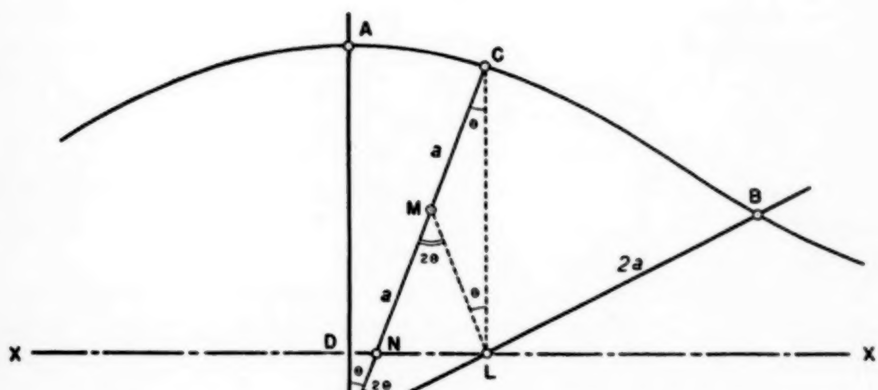


FIG. 5

Unlike the Quadratrix, which, once drawn, can be used to trisect any given angle immediately, a fresh Conchoid must be constructed for each new angle. Suppose it is required to trisect AOB, Fig. 5. Place the angle with vertex at O and draw the perpendicular line XX, cutting OB at L so that $OL = a$, the projection value: cos(AOB). Using $2a$ as the radius of the generating circle, draw the Conchoid ACB. At L construct the parallel to OA which will meet the curve at C. The line OC then trisects AOB. The proof of this is direct:

Let angle $AOC = LCO = \theta$. Now since $CN = 2a$ (by definition of the curve) and CLN is a right angle, then the segment from the mid-

point M of CN to the right-angled vertex L is of length a. Thus triangles CML and MLO are both isosceles. Accordingly,

$$\angle AOC = \angle OCL = \angle MLC = \Theta.$$

But $\angle OML = 2\Theta$, since it is the exterior angle of triangle CML. Thus

$$\angle MOL = 2\Theta$$

and the angle AOB is trisected by OC.

A *polar* equation of the curve may be derived directly from the definition by selecting OA in Fig. 5 as the polar axis and O as the pole. We have $OC = r$ and angle $AOC = \Theta$, where r and Θ are now understood to be variables. If we denote the distance OD by b, we have from the right triangle ODN:

$$\cos \Theta = b/ON \quad \text{or} \quad ON = b/\cos \Theta.$$

Thus
$$\boxed{r = b/\cos \Theta + 2a}$$

is the polar equation of the upper branch.

Using XX and OA as X and Y axes respectively, the rectangular equation results from substituting $r = \sqrt{(x^2+y^2)}$ and $\cos \Theta = y/\sqrt{(x^2+y^2)}$ in the preceding polar equation. We find, after squaring:

$$\boxed{(x^2+y^2)(y-b)^2 = 4a^2y^2}\ .$$

(The reader will find this interesting shell-shaped curve quite easy to construct. If 60° be the given angle, the corresponding Conchoid has $a = \cos 60° = 1/2$ and $b = a \cdot \cos 60° = 1/4$).

A word of caution should be made here against a possible misunderstanding. Although points on the curve may be found by straightedge and compasses, the *continuous* description is entirely beyond the possibilities of these instruments.

3. *The Hyperbola*

In solving the Trisection Equation, Pappus, about 300 A. D., made use of some properties of conic sections that were well known at that time. His method of trisection [**39**]* is essentially this: a unit circle is described with center at the vertex of the given angle AOB and the bisector OC constructed. A point P is allowed to move so that its distance from B is always *twice* its distance from the bisector OC. In this fashion P traces out a branch of an Hyperbola with the line

*Such bracketed numbers refer to items in the Bibliography to be attached to the last chapter.

OC as the *directrix* and the point *B* as *focus*. This branch is reflected in *OC* so that P' corresponds to *P*.

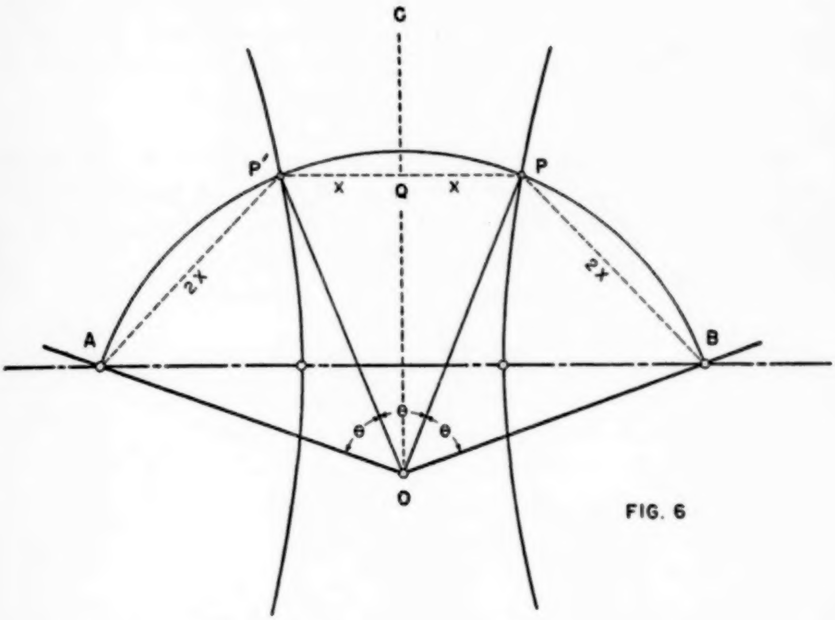


FIG. 6

The points of intersection of the unit circle and the Hyperbola are trisecting points of the arc $AP'PB$. For, if $PQ=x=P'Q$, then $PB=P'A=2x$ and the three isosceles triangles AOP', $P'OP$, and POB are congruent to each other with equal angles at their common vertex *O*.

To derive the rectangular equation of the curve, let *AB* and *OC* represent the *X* and *Y* axes. If we denote by $2c$ the distance *AB*, then *B* has the coordinates $(c,0)$ and we need only express in symbols the requirement that the distance *PB* must at all times be twice the distance *PQ*. That is, if the coordinates of *P* are (x,y):

$$\sqrt{(x-c)^2+y^2}=2x$$

or

$$\boxed{y^2-3x^2-2cx+c^2=0.}$$

The location of the trisecting point *P* requires the simultaneous solution of this equation and that of the circle, which is itself of the second degree. This gives rise to an equation of the fourth degree, the roots of which are the coordinates of *P* together with those of trisecting points for induced angles.

4. *The Limaçon*

The Limaçon, invented by Pascal about 1650, was later found to have trisecting possibilities. It is defined in a manner similar to the Conchoid: A point F is selected upon a fixed circle of unit radius. The movable line FA, which passes always through F, intersects the circle at P. The point A on the line at a constant distance b from P describes the curve.

The polar equation of the curve may be derived by taking the diameter FB as polar axis and F as pole. A then will have the coordinates (r,θ). Angle FPB is inscribed to the semicircle and is accordingly a right angle. Thus $FP=2\cos\theta$. Directly then,

$$\boxed{r=2\cos\theta+b}$$

is the equation of the path of A. Replacing r by $\sqrt{(x^2+y^2)}$ and $\cos\theta$ by $x/\sqrt{(x^2+y^2)}$ produces the rectangular equation:

$$\boxed{(x^2+y^2-2x)^2=b^2(x^2+y^2).}$$

Thus the Limaçon is a curve of the fourth degree.

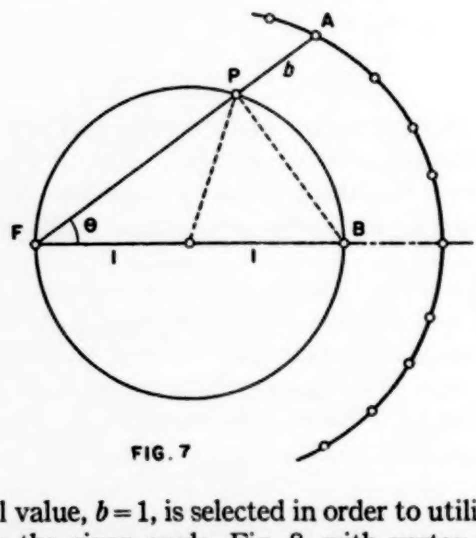


FIG. 7

The special value, $b=1$, is selected in order to utilize the curve as a trisector. Place the given angle, Fig. 8, with vertex at O, the center of the unit circle, and one side along its diameter FB. The other side will strike the Limaçon at A. Draw AF. Then the line through O parallel to AF trisects AOB. The proof follows: We have by construction:

$$AP=PO=FO=1,$$

so that triangles FOP and OPA are isosceles. Thus, if angle $OFP = \Theta$,

$$\angle OPF = \Theta \quad \text{and} \quad \angle POA = \angle PAO = \Theta/2.$$

But $\angle BOP = 2\Theta$ since it is the exterior angle of triangle OFP.

Accordingly, $\angle AOB = \angle BOP - \angle AOP = 2\Theta - \Theta/2 = 3\Theta/2$,

and thus $$\angle PAO = \angle AOB/3.$$

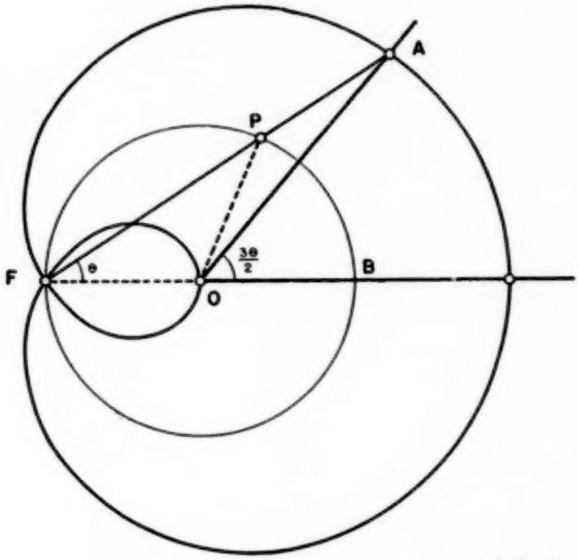


FIG. 8

There are three types of Limaçon which will be interesting to sketch by taking, for example, $b=1$, $b=2$, $b=3$. One will appear with the loop, one without, and one with a *cusp*. This last, the heart-shaped *Cardioid*, has many beautiful properties. It can be generated by a point on the rim of a circular disk rolling on another of equal size and also can be seen as the curve of light rays reflected from a polished cylinder.

5. *The Parabola*

Rene Descartes, called the founder of modern Analytic Geometry, published the epoch-making treatise [12] "La Geometrie" in 1637. Contained in this monumental work is another attack upon the Trisection Problem, a solution by means of conic sections. The idea involved is that the roots of the Trisection Equation

$$x^3 - 3x - 2a = 0$$

can be represented as the x-coordinates of the points of intersection of a Parabola and a circle. Consider the Parabola:

$$y = x^2$$

and the circle: $\qquad x^2 + y^2 - 2hx - 2ky = 0,$

whose center is at the point (h,k). The abscissas of their points of inter-

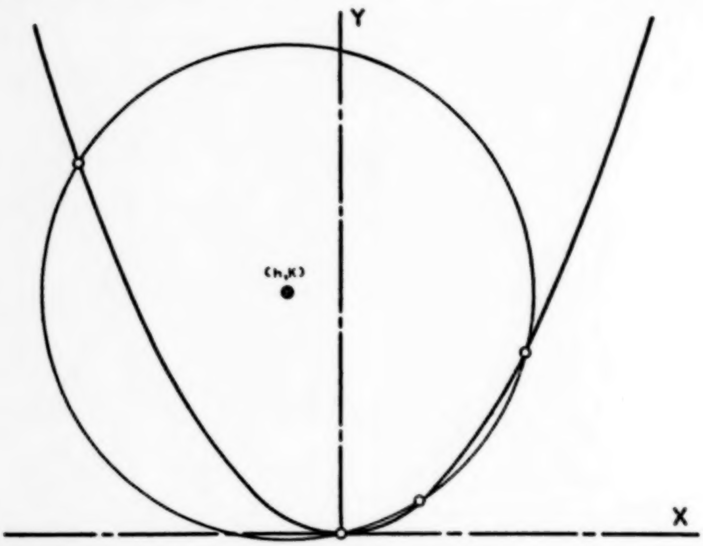


FIG. 9

section are found by eliminating y between their equations; thus, by substitution:

$$x^2 + x^4 - 2hx - 2kx^2 = x[x^3 - (2k-1)x - 2h] = 0.$$

The factor $x = 0$ which was expected since both curves pass through the origin, may be discarded. The other factor:

$$x^3 - (2k-1)x - 2h = 0$$

can be the given Trisection Equation if we take particular values for h and k; that is, if

$$2k - 1 = 3 \quad \text{or} \quad k = 2 \quad \text{and} \quad h = a.$$

This is the requirement that the center of the circle be taken at $(a,2)$. With such arrangement, the circle will cut the Parabola in points whose abscissas are the roots of the Trisection Equation.

For illustration, let us apply this method to the trisection of 60°, the corresponding Equation for which is

$$x^3 - 3x - 1 = 0.$$

Construct the Parabola $y = x^2$. Then draw the circle whose center is (1/2,2) and which passes through the origin O. The x-coordinate OC of a point of intersection is shown in Fig. 10.

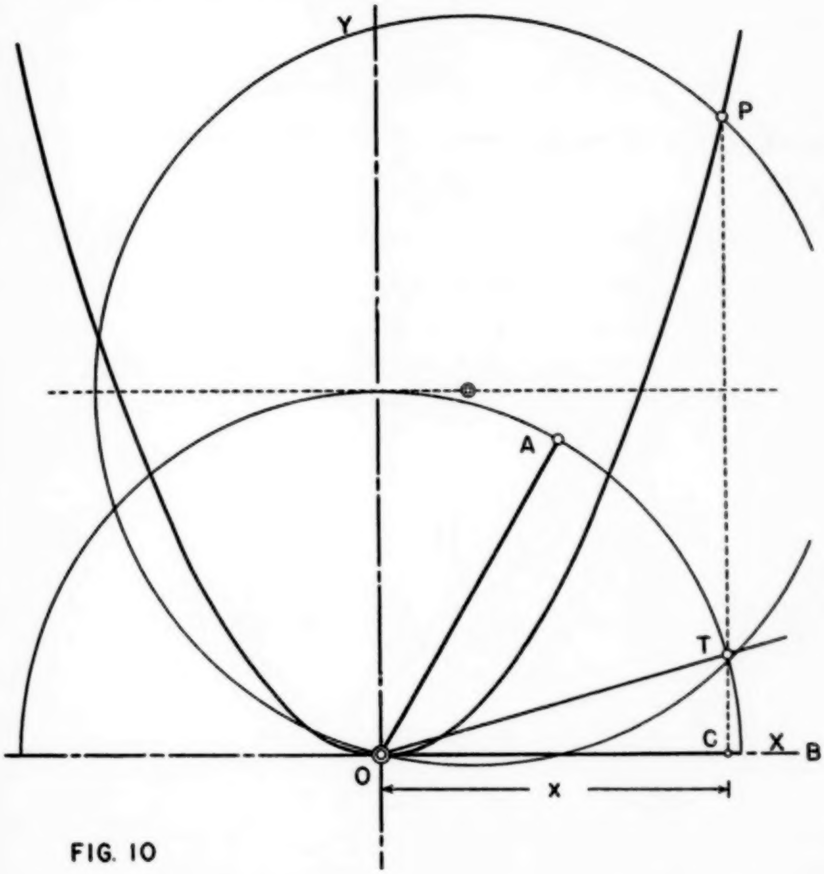


FIG. 10

It is obvious that the Parabola can be drawn once for all angles. When given any particular angle such as AOB (with OA selected as 2 units), drop the perpendicular from A to OB. Halve this projection and erect another perpendicular to meet the line $y = 2$ at the center of the required circle. Draw the circle passing through the origin. From the point P, where this circle meets the Parabola, drop the perpendicular to OB. This will determine the root x of the Trisection Equation.

But this value, see Fig. 1, is *twice* the projection value of the trisected part of AOB. That is, $x/2 = \cos \Theta$. Thus we may either halve x and erect the perpendicular to meet the unit circle or, more conveniently, draw the circle with radius 2 meeting PC in T. The line OT then is a trisector of AOB.

Notice that the circle has for radius the quantity: $\sqrt{(2^2+a^2)}$. Thus, since the numerical value of a is never greater than 1, the largest radius of any circle needed is $\sqrt{5}$. For this reason, the Parabola need not be drawn beyond a certain range.

6. *The Cubic Parabola*

The curve whose equation is $y = x^3/2$ cuts the line

$$y = 3x/2 + a \tag{6.1}$$

in the points whose x-coordinates are given by the cubic equation:

$$x^3/2 - 3x/2 + a = 0 \qquad \text{or} \qquad x^3 - 3x - 2a = 0.$$

This system then may be used for trisection:

To each given angle with its projection value a there will correspond a certain line (6.1). All such lines have the same *slope:* 3/2;

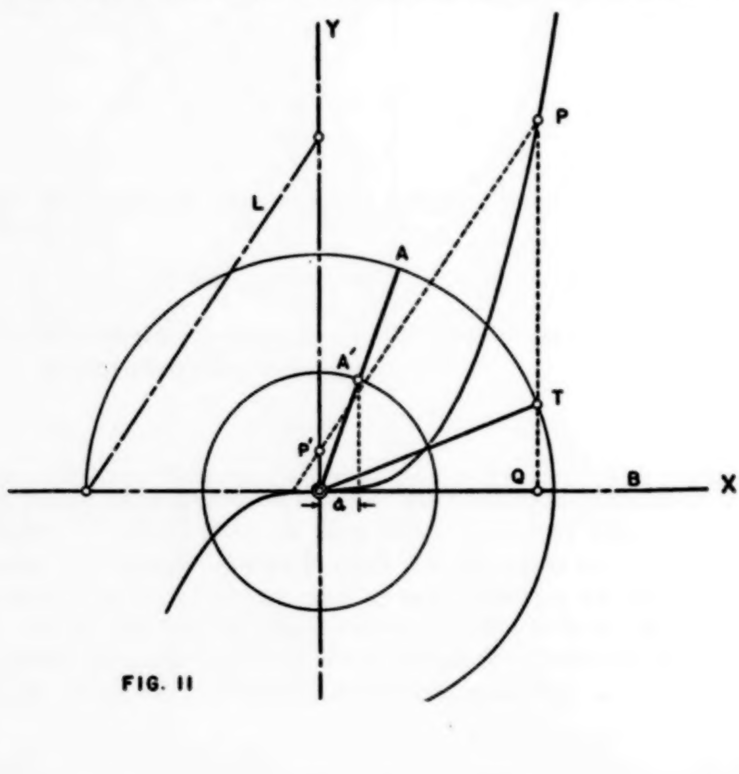


FIG. 11

that is, they are all parallel to the segment L drawn in Fig. 11. Furthermore, the line corresponding to any given angle AOB cuts off upon the Y-axis a segment equal to a itself. The geometrical construction for the trisection of AOB is thus indicated: Draw the circles of radii 1 and 2 as shown. From A', where OA meets the unit circle, drop the perpendicular to find the projection a. Lay off this projection length $OP'=a$ upon the Y-axis and draw the line PP' parallel to L. From the intersection point P drop the perpendicular to OB, thus determining $OQ=x$, a root of the Trisection Equation. Now, as explained in the preceding paragraph, T is a point on the trisector.*

7. *The Cycloid of Ceva*

Prompted by the familiar "insertion" method (see Chapter III) of Archimedes, Ceva devised in 1699 a curve for trisection which was called the "Cycloidum anomalarum". The principle involved is that of doubling angles. With center C on the fixed line CB, draw the unit circle. A point P on a line rotating through C is located so that

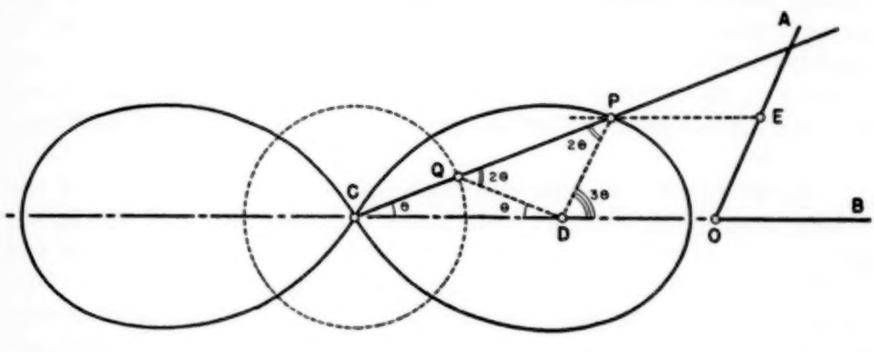


FIG. 12

$$CQ=QD=DP=1.$$

The locus of P, as AC revolves about C, is the curve in question. It is evident from the figure that if angle $QCD=\theta$, then

$$\angle QDC=\theta \quad \text{and} \quad \angle PQD=\angle QPD=\angle 2\theta.$$

Now since $\quad \angle QDP=\pi-4\theta \quad \text{and} \quad \angle QDC=\theta,$

then $\quad \angle PDO=3\theta \quad \text{and} \quad \angle QCD=\angle PDO/3.$

*Points P in the first quadrant determine trisectors of given acute angles, while the other intersections in the third quadrant locate trisectors for the "induced" angles.

The application to a given angle is as follows. Place one side, OB, of the angle coincident with the line CB. With the compasses lay off the unit length OE on OA. Then draw the line EP parallel to CB which strikes the curve at P. Then $\angle PCD = \angle AOB/3$.*

The equation of the curve in polar coordinates is:

$$\boxed{r = 1 + 2\cos 2\theta},$$

and in rectangular coordinates:

$$\boxed{(x^2+y^2)^3 = (3x^2-y^2)^2}.$$

The Cycloidum anomalarum is then a curve of the sixth degree. Compare the polar equation of this curve with that of the Limaçon.

8. *Remarks*

We found in Chapter I that the cubic Trisection Equation could not be solved by means of the first degree equation of a line and the second degree equation of the circle except when the Equation had constructible roots. In this chapter we have presented a number of solutions of the Trisection Equation in its general form, that is, for any value of a, but in each instance we made use of equations and corresponding curves which, excepting the conics, were of higher degree than the second.

Mathematical literature is crammed with such solutions of the Trisection Problem as are given in this chapter. It is an interesting fact that there exists an infinitude of curves, both transcendental and algebraic, which furnish the means of solving the problem. These curves, for the most part, are difficult to draw. Mechanical devices of various sorts have been invented for the description of these higher plane curves and, in many instances, these instruments may be used as trisectors in direct fashion. This is the subject of the following chapter.

*Lines EP cut the loop on both sides of its highest point. Those intersections to the right determine trisectors for acute angles while those to the left give trisectors for obtuse angles.

Professor O. W. Albert, University of Redlands, and Professor Sophia H. Levy, University of California, have served as a nominating committee for the election of Governor of the M. A. A. from the States of California and Nevada. They have nominated Professor H. M. Bacon, Stanford University, for this post. Professors Albert and Levy are chairmen of the Southern California and Northern California sections of the M. A. A.

—Reported by L. J. Adams.

Mathematical World News

Edited by
L. J. ADAMS

The first regular meeting of the Saskatchewan Mathematical Association was held on September 28, 1940, at the University of Saskatchewan. More than thirty persons attended the meeting, including the following twenty-nine members: C. E. Behrens, G. M. Busche, L. M. Chapman, N. K. Cram, Greta Denison, D. Derry, W. L. Eddy, H. H. Ferns, A. R. Frith, G. Herzberg, Rev. Paul Kuehne, Ester Lamb, D. S. McArthur, F. J. MacDonald, R. J. Mathers, R. L. Mœnter, C. A. Morgan, D. C. Murdock, Rev. Augustine Nenzel, A. J. Pyke, L. F. Smith, W. R. Shanklin, K. G. Towes, V. Valgarsson, C. J. Hewitt, Alma E. Hiebert, F. M. Holmes, R. D. James, K. Korven. After an address of welcome by President Thomson, papers were presented by Prof. N. B. Hutcheon and Prof. R. D. James:

1. *Mathematics in Engineering.* Professor N. B. Hutcheon.
2. *The Four Cube Problem.* Professor R. D. James.

The following is a resumé of some of the remarks by Professor James:

A problem which has puzzled mathematicians for a long time is whether or not every positive integer can be represented as a sum of four cubes of integers. For example:

$$24 = 5^3 + 3^3 + (-4)^3 + (-4)^3.$$

The identity of which the above example is a special case is:

$$6K = (K-1)^3 + (K-1)^3 + (-K)^3 + (-K)^3.$$

This shows that every integer of the form $6K$ may be written in the form:

$$x^3 + y^3 + 2z^3, \tag{1}$$

where x, y, and z are positive or negative integers. Now it has been verified that every positive integer less than 100 whether of the 10*rm* $6K$ or not may be written in the form of equation (1) with the possible exception of the integers 76 and 99, for which a solution has not yet been found. It is suggested that an interesting arithmetical recreation would be the extension of the representation of integers in the form of equation (1) to integers greater than 100. It would also be interesting

to try to find the representation of the missing integers 76 and 99 in the form of equation (1).

The University of Toronto Press announces a new series of books under the title *Mathematical Expositions*. The first volume is: *The Foundations of Geometry* by Professor Gilbert de B. Robinson of the University of Toronto.

The American Association for the Advancement of Science, Section A, was scheduled to meet at the University of Pennsylvania on December 27, 1940. Two addresses scheduled were:

1. *A Mathematical theory of equilibrium with application to minimal surfaces.* Professor Marston Morse, retiring vice-president of the A. A. A. S. and chairman of Section A.
2. *The mathematical problems in meteorology.* Dr. C. G. Rossby, United States Weather Bureau.

Professor George Rutledge, head of the mathematics department at Massachusetts Institute of Technology, died September 21, 1940, at his home in Belmont, Massachusetts. An alumnus of the University of Illinois, he was a professor at Georgia School of Technology before going to Massachusetts Tech in 1915. He was born at Jacksonville, Illinois, in 1881, and attended Whipple Academy there before entering the University of Illinois. He was a member of Sigma Xi. His son Philip teaches at Purdue.

The mathematics section of the Wisconsin Education Association met on November 7, 1940. The program included a discussion luncheon and two addresses:

1. *We, Mathematically Trained Citizens.* Professor Mark H. Ingraham, University of Wisconsin.
2. *Mathematics for All Laymen.* Harold Fawcett, University High School, Columbus, Ohio.

Discussion leaders included: Ethelwynn R. Bechwith, R. O. Christoffersen, Ethel Daley, H. P. Evans, Louise Fromm, Alice Grueberger, Mary E. Henry, Oscar Melby, Elda Merton, Vernon Ramberg, J. H. Rose, Elmer G. Schuld, Victor Schumann, A. J. Smith, Margaret Striegl, Orpha Thompson, Paul W. Waterman, Theodore C. Potter.

The celebration of the fiftieth anniversary of the University of Chicago will take place during 1941. Some thirty-one societies will hold meetings at the University, among which are the Mathematical Association of America, the American Mathematical Society and the Algebra Conference.

Problem Department

Edited by
ROBERT C. YATES and EMORY P. STARKE

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to ROBERT C. YATES, Mathematics, University, Louisiana.

SOLUTIONS

No. 364. Proposed by *C. W. Trigg*, Los Angeles City College.

Through any point on the longer diagonal of a rhombus, which is formed from two equilateral triangles, lines are drawn parallel to the sides.

(1) The four intercepts and the extremities of the shorter diagonal are concyclic.

(2) The six points determine two congruent equilateral triangles with the same centroid.

(3) Find the radius of the circumcircle in terms of the segments of the diagonal.

Solution by *D. L. MacKay*, Evander Childs High School, New York.

Let P be any point on the longer diagonal AC of the rhombus $ABCD$, one of whose angles is 60 degrees. The line EPF, parallel to BC, cuts AB in E and CD in F, and GPH, parallel to AB, cuts AD in G and BC in H.

(1) Since $EP \times PF = GP \times PH$, points E, G, H, and F are concyclic; and since angles EBH and GDF are supplements of angles EGP and GEP respectively, B and D lie on this circle.

(2) If GH and EF cut BD in K and L respectively, then from the congruency of triangles EBH, EGD, DKH, GDF, BFL, GEB, the triangles DEH and BFG are congruent equilateral

triangles. Since the latter are inscribed in the same circle their centroid is the center of the circle in (1).

(3) Let $AB=2a$, $AP=2b$. Then $AC=2a\sqrt{3}$, $PC=2a\sqrt{3}-2b$, $AE=\frac{2}{3}b\sqrt{3}$. From $ED^2=AE^2+AD^2-2AE\cdot AD\cos 60°$, we have

$$9R^2=12a^2-4ab\sqrt{3}+4b^2=(2a\sqrt{3}-2b)^2+4ab\sqrt{3}=PC^2+AP\cdot AC$$

$$R=\tfrac{1}{3}\sqrt{AP^2+AP\cdot PC+PC^2}$$

Also solved by *Walter B. Clarke* and the *Proposer*.

No. 367. Proposed by *Dewey C. Duncan*, Los Angeles City College.

Find all rectangular parallelopipeds whose principal diagonals are 49 units in length and whose edges are all of integral lengths.

Solution by *Walter B. Clarke*, San Jose, California.

This problem is equivalent to a solution in positive integers of

$$x^2+y^2+z^2=49^2=2401.$$

With $x\leqq y\leqq z$, we must have $28<z<49$. Substitution of these twenty values of z in $2401-z^2$ gives integers which must be of the form x^2+y^2. Nine of these are immediately discarded because a sum of two squares must be divisible by 3^2 if by 3. Others are eliminated by the analogous property of 11, 19 and 47 (any prime of the form $4n-1$). If the remaining values are expressed as sums of two squares in every possible way, there result eight solutions, viz. $(x, y, z)=(4, 9, 48)$, $(14, 21, 42)$, $(12, 24, 41)$, $(15, 24, 40)$, $(4, 33, 36)$, $(9, 32, 36)$, $(12, 31, 36)$, and $(23, 24, 36)$.

No. 368. Proposed by *F. C. Gentry*, Louisiana Polytechnic Institute.

If A, B, C and a, b, c, are respectively the angles and the lengths of the opposite sides of a triangle then

$$\begin{vmatrix} a\cos^2 A & b\cos^2 B & c\cos^2 C \\ \cos A & \cos B & \cos C \\ a & b & c \end{vmatrix} = 0.$$

Solution by *W. L. Roberts*, student, Colgate University.

If the determinant D is expanded in terms of the elements of the second row, we get for the first of the three terms

$$P_A=-bc\cos A(\cos^2 B-\cos^2 C).$$

By simple trigonometric relations this term becomes

$$P_A = bc \sin A \cos A \sin(B-C),$$

and upon use of the law of sines can be written in the form

$$P_A = abck \cos A \sin(B-C).$$

By cyclic change of the symbols involved the other two terms of the expansion, P_B and P_C are

$$P_B = abck \cos B \sin(C-A) \quad \text{and}$$

$$P_C = abck \cos C \sin(A-B).$$

Summing up the three terms and performing the expansions indicated, it appears at once that $D=0$, which is the desired result.

Also solved by *D. L. MacKay*, *Jean Andres Thomas*, and the *Proposer*.

No. 370. Proposed by *N. A. Court*, University of Oklahoma.

The vertices of a variable tetrahedron with a fixed circumcenter lie on two fixed skew lines. (1) Show that the centroids of the faces lie on two fixed straight lines. (2) May the tetrahedron in some of its positions become isosceles (i. e., may each edge become equal to the respectively opposite edge)?

Solution by *Paul D. Thomas*, Norman, Oklahoma.

(1) Let the given skew lines be k and k' and O the fixed circumcenter. Drop perpendiculars from O upon the lines k and k' meeting them respectively in H and H'. Then HH' is a fixed bimedian for all the tetrahedrons of the system. Let $ABCD$ be one position of the variable tetrahedron, AB on k and CD on k'. (H must be the midpoint of AB, H' the midpoint of CD, and $OA=OB=OC=OD$.) Let E, F, G, M be the centroids of the faces ABC, ABD, ACD, BCD, respectively. Then $BM=2(BH')/3$, $AG=2(AH')/3$. Hence MG is parallel to AB. GM and HH' are coplanar and meet in a point S. But $HS=2(HH')/3$, and since HH' is fixed, the point S is a fixed point. Therefore GM is fixed, since it is parallel to the fixed line k and passes through the fixed point S on the fixed bimedian HH'. Similarly it may be proved that EF is parallel to k' and passes through a fixed point on HH'.

(2) In an isosceles tetrahedron the bimedians coincide with the bialtitudes and the circumcenter coincides with the centroid.* In any

*N. A. Court, *Modern Pure Solid Geometry*, p. 95.

tetrahedron the bimedians bisect each other in the centroid.* Hence, if a tetrahedron is to be isosceles the fixed circumcenter must be the midpoint of the common perpendicular of the two given skew lines.

No. 371. Proposed by *C. C. Chaudoir*, Baker, Louisiana.

Prove: In a right triangle whose sides are integers without common divisor, if a leg B is a power of 2 then the other leg and the perimeter are each divisible by $\frac{1}{2}(B+2)$.

Solution by R. V. Sweeney, student, Colgate University.

It is known that, for any two positive integers, m and n, without common divisor, not both odd, and with $m>n$, the numbers m^2-n^2, $2mn$ and m^2+n^2 represent the sides of a right triangle without a common divisor; all such right triangles are thus given. Under these conditions, the only leg that can be a power of 2 is the one represented by $2mn$. Therefore we have

$$B=2mn=2^k, \quad mn=2^{k-1},$$

and, since m and n are not both odd and $m>n$,

$$m=2^{k-1} \quad \text{and} \quad n=1.$$

It follows that the other leg A and the perimeter P have the values

$$A=2^{2k-1}-1 \quad \text{and} \quad P=2^k(2^{k-1}+1).$$

It is then evident that each of these is divisible by $\frac{1}{2}(B+2)=2^{k-1}+1$.

Also solved by *Walter B. Clarke*.

PROPOSALS

No. 381. Proposed by ***Paul D. Thomas***, Norman, Oklahoma.

The locus of the point Q is a circle if Q is the foot of the perpendicular from a point P upon the polar of P with respect to the conic $x^2/t+y^2/b^2=1$, where t is a parameter.

No. 382. Proposed by *D. L. MacKay*, Evander Childs High School, New York.

Given line AB and circles O and O' on opposite sides of AB, construct an equilateral triangle having a vertex on AB and on each of the given circles. A solution involving the method of symmetry is desired.

**Ibid.*. p. 48.

No. 383. Proposed by *E. P. Starke,* Rutgers University.

Find the number of triangles of all kinds whose sides are positive integers and whose largest side does not exceed a given number, K.

No. 384. Proposed by *W. L. Roberts,* Colgate University.

In an attempt to locate an enemy cannon during the World War, microphones were stationed on a straight line at A, at B, 2,200 feet from A, and at C, 4,400 feet from A. The explosion reached B $\frac{1}{2}$ second and C $1\frac{1}{2}$ seconds after it was heard at A. Assuming the speed of sound to be 1,100 feet per second, find how far the enemy cannon was from points A, B, and C, and at what angles with the line ABC, cannon must be adjusted at A, B, and C to destroy the enemy gun.

No. 385. Proposed by *D. L. MacKay,* Evander Childs High School, New York.

The medians of triangle ABC intersect its circumcircle in A', B', C'. Given these three points, construct the triangle ABC.

No. 386. Proposed by *N. A. Court,* University of Oklahoma.

In a given plane to find a point such that its harmonic plane for a given tetrahedron shall pass through the harmonic pole of the given plane for the same tetrahedron.

Emergency courses to train engineers and technicians needed in the nation's defense industries will be offered soon in a program to be sponsored jointly by the Massachusetts Institute of Technology, Harvard University, Northwestern University and Taft's College. Information may be obtained from the Engineering Defense Training Bureau, Room 7-102, Massachusetts Institute of Technology, Cambridge, Massachusetts.

The celebration of the fiftieth anniversary of the University of Chicago will take place during 1941. Some thirty-one societies will hold meetings at the University, among which are the Mathematical Association of America, the American Mathematical Society and the Algebra Conference.

Under the joint auspices of Princeton University and the Institute for Advanced Study the Princeton University Press publishes the *Annals of Mathematics,* the *Princeton Mathematical Series,* and the *Annals of Mathematics Studies.*

—Reported by L. J. Adams.

Bibliography and Reviews

Edited by
H. A. SIMMONS

Calculus. By Charles K. Robbins and Neil Little. Macmillan, New York, 1940, vi+398 pages. $3.25.

The book to be reviewed is quite unique among elementary mathematics texts because it contains almost no valid proof or correct definition. A proof in this book usually begins with an example which is worked out somehow. About half way through the proof there appears a statement "it can be shown", and the conclusion is written once in light type and once in heavy type. In some of the proofs there is not even that much of a concession to decency. Definitions are rarely stated; they are usually relegated to foot notes which do not explain or define. The first half dozen chapters are based on a "dictionary definition of limits" (cf. footnote p. 2), and with no more of a tool than this we obtain the dx^n/dx by means of the binomial theorem. The jump from the limit of a sum of terms containing $\Delta x, \Delta x^2$ etc. to the final result is accomplished without even raising a question as to the limit of a product or a sum. True, a little later, in finding the derivative of a sum of two functions a footnote does mention that the limit of a sum is the sum of the limits. In deriving the derivative of the logarithm of a function, one arrives at

$$\lim_{\Delta x\to 0}\left(1+\frac{\Delta u}{u}\right)^{u/\Delta u}$$

and here again, "it can be shown that the limit is $e=2.718\ldots$".

Definite integrals and partial derivatives are treated with even less respect and consideration. There is not even any mention throughout the book of the fact that

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}$$

under certain restrictions. Somewhat belatedly (Ch. VIII) there appears a definition of the limit of a variable (not of a function, although that is what the authors use), with the old refrain "becomes and remains" that mathematicians have discarded quite a long while before this book was written. In double integration "$\Delta x\Delta y$ is substantially a point with coordinates (x,y)(cf. p. 296)" and the theory of maxima and minima of functions of several variables is covered in a dozen lines.

In differentiating implicit functions nothing is said about the validity of the process, but there are the words of guidance! (cf. p. 133) "it is only necessary that the student note in the following examples how the results are simplified." There ought to be a law!

In reviewing a book it is customary to praise something about it, and in this case the reviewer can praise unreservedly the typography and general makeup of the book which are very good. The type is extremely clear; the illustrations are excellent. There are also numerous examples throughout the book and there is even a whole chapter, V, of examples on maxima and minima of functions of one variable.

Washington State College. M. S. KNEBELMAN.

Plane and Spherical Trigonometry. By Lyman M. Kells, Willis F. Kern, and James R. Bland. McGraw-Hill Book Company, Inc., New York, 1940. 15+401 pages. \$2.00.

This text is a revision of the authors' *Plane and Spherical Trigonometry* published five years ago. The experience gained from actual class room use of the first edition together with numerous constructive criticisms enabled the authors to write a text definitely better than the first edition.

The fundamental definitions are introduced and explained by means of simple numerical right-triangle problems within the range of the student's experience. First the acute right triangle is studied. There are even many identities to be proved true for acute angles only. There is little unusual in the presentation of theory. The importance of each topic is stressed before presenting it. The outstanding feature of the text lies in its numerous applied problems. Many of these are concerned with ships, airplanes, etc. A realistic picture, not a schematic drawing, adds interest. As long as the present emphasis on preparedness continues the military setting of numerous problems will no doubt interest the students.

This text is carefully written and covers the subject rather fully. It is not a brief text. It will undoubtedly be widely used.

DePaul University. JOHN J. CORLISS.

Metric Differential Geometry of Curves and Surfaces. By E. P. Lane, University of Chicago Press, Chicago, 1940. viii+216 pages. \$3.00.

The first two chapters of this book are devoted to the theory of curves (curvature, torsion, the Frenet formulas, the moving trihedron, and applications). The remaining four chapters introduce the reader to the theory of surfaces (the two fundamental forms, curves traced on surfaces, the equations of Wingarten and Gauss, curvature, and transformations of surface). A short bibliography lists the treatises of Bianchi, Blaschke, Cesaro, Darboux, Eisenhart and Scheffers.

The treatment is elementary; it requires only analytic geometry and a first course in calculus. The statement is made by the author that "The advantages of a treatment by means of vectors are well known, but it has been thought best, in order to make the discussion as elementary as possible, to refrain from employing them here." The choice of the method is a matter of taste, but the question arises whether the vector method by its appeal to intuition would not have provided a more effective elementary approach to the subject. In fact there have been many analytic geometries published in French during the past fifteen years with a moderate appeal to vector analysis, as nothing more than scalar product, vector product, and the derivative of a vector are needed. It is to be noted that left-handed coordinate systems are used consistently throughout the book: that would call for some precautions is using other reference books.

Virginia Military Institute. W. E. BYRNE.

An Introduction to Abstract Algebra. By Cyrus Colton MacDuffee. John Wiley and Sons, New York, 1940. vii+303 pages.

This book is intended to be used as a textbook by beginning graduate students. It will admirably serve this purpose, because it is very carefully written and is an introduction in fact as well as in name. The abstract point of view is developed grad-

ually, as the author states in the preface. Concrete instances are presented before an abstract idea is developed. The first four chapters are devoted to the elements of the theory of numbers, of the theory of finite groups, of the theory of algebraic fields and integral algebraic domains. With this background, the student will appreciate more easily the abstract developments in the remaining four chapters, on rings and fields, matrices, and linear associative algebras.

Certain fundamental concepts are very clearly presented, notably those of homomorphic mapping, of direct product of two finite groups, and of the polynomial ring over a general ring. There is a desirable emphasis on an equals relation as a part of a mathematical system. However, it would have been well to retain the standard term equivalence relation, and to emphasize the idea of the classification which is determined by, and determines, an equivalence relation. For example, the important concept of quotient group is much more simple when this idea of classification is used on page 62 than it is at first, in Theorem 26.1, when merely an equals relation is used. Again, the idea of classification would have clarified the development of the algebraic extension of a field in §78.

The existence of infinite groups could have been emphasized, and many properties developed without further details, if the word *finite* had been inserted in the chapter on groups only when this hypothesis was actually used. Certainly, if the word *finite* was inserted in some of the theorems, it should have been inserted in all of the theorems in which the hypothesis *finite* was used.

The Galois theory for normal fields over a general field could well have been developed, especially in view of the detailed treatment for algebraic fields.

In the chapter on matrices, amplification of the solution of simultaneous linear equations as an application of the theory of linear systems of vectors would have been desirable. The proof that a matrix satisfies its characteristic equation, with the preliminary clarification, is commended, if this method of proof is adopted at all.

In the chapter on the theory of numbers there are two unfortunate attempts at generalization. The first is in permitting, by definition, a prime integer to be negative. The second is in permitting the modolus m to be negative in the definition of residue classes and the Euler φ-function. Then certain statements and proofs in the following pages are incorrect: for instance, in Theorems 9.2, 10.2, 10.3, 11.3, in Corollary 11.1, in problem 1 on page 11, etc.

Certain unusual notations occur. For example, z is used instead of 0 for the unit element of the additive group of a ring. Again, if B is a matrix, then B^A is used for the adjoint of B. and B^T for the transpose of B. Again, Ra denotes the rational field, and Ri a general ring with unit element i. Misprints were noted on pages 119 and 243.

Northwestern Universoty. L. W. GRIFFITHS.

Mathematics and the Imagination. By Edward Kasner and James Newman. (with drawings and diagrams by Rufus Isaacs.) Simon & Schuster, Inc., New York, 1940. xv+380 pages. Cloth $2.75.

This volume has already taken its place among the current best sellers. Apparently it has succeeded in communicating to the layman something of the pleasure experienced by the creative mathematician in difficult problem solving. It is no mean feat thus to combine mathematics with the human imagination and so transport the individual reader beyond common experience and ordinary intuition. The authors have aimed to extend this process of *haute vulgarisation* to the outposts of higher mathematics, usually referred to only by name among laymen. They feel that mathematics, in large measure, remains unrevealed.

Following a discussion of the "new language" of mathematics, a large number called the *googolplex* is introduced. Another aspect of the book is the inclusion of puzzle and recreational material, in order to stimulate mathematical thinking on the part of the reader, as well as to illustrate historically the importance of some of the topics considered. Various types of geometry come in for much attention. The use of the transcendentals π and e, as well as $\sqrt{-1}$, are also explained. One valuable feature of the text is the pleasurable introduction to problems of topology. Probability and chance have always appealed to the popular mind and hence find an extended treatment here.

Paradoxes and fallacies are taken up in such a way as to remove some of the current confusion along this line. Elementary concepts of the calculus are put forward, and the reviewer feels that more of this style in presenting such material should be included in our textbooks.

The epilogue bears the same title as the book itself, and attempts to answer the question: What is mathematics? It is a "universal language" (p. 358). "We have overcome the notion that mathematical truths have an existence independent and apart from our own minds" (the environment) (p. 359). "Some of the landmarks are fixed; some of the vast network of roads is made understandable; there are guideposts for the bewildered traveler.... We share the feeling that mathematics is more than a factory of tautologies, rather that it is a vehicle to carry on the highest aspirations of the creative intellect.... Yet ultimately mathematics reaches pinnacles as high as those attained by the imagination in its most daring reconnoiters.... For in their prosaic plodding both logic and mathematics often outstrip their advance guard and show that the world of pure reason is stranger than the world of pure fancy" (pp. 360,362).

The format and typography are excellent; there is a five page bibliography and also an index.

State Teachers College, LaCrosse, Wisconsin. G. WALDO DUNNINGTON.

Bibliography of Mathematical Works Printed in America Through 1850. By L. C. Karpinski, University of Michigan Press, 1940. xxvi+679 pages. $6.00.

Professor Karpinski has rightly dedicated the present monumental work to American Librarians and Libraries. He states in the Introduction that the dissemination of mathematical knowledge is an integral part of the development of science in particular, and knowledge in general in the United States. It is the purpose of this volume to trace the development of the rise of mathematical knowledge in the United States, Canada, Central and South America by tracing the development of the publishing of textbooks and periodicals in mathematics.

Before presenting the bibliography proper the author gives a short background of the British influence on mathematical studies in the colonies. Another important feature is the recognition of the Spanish influence on the early mathematical development in the Americas. He points out that "Almost without exception the American-made Spanish texts of Central America were far more scientific and far better adapted to local American conditions and problems than were the textbooks of the English Colonies."

The scope of the work proper is rather inclusive. The first section consists of the "Bibliographical List of Books, Pamphlets and Periodicals" given in the order of their publication, beginning with 1556 and continuing through 1850. In this group are listed all those titles known to have been published within these years. The second section carries the "Entry Titles," or those works whose publication is problematical. This

section is composed of those entries for which title-pages were actually printed but for which there was no subsequent completion of copyright by the recorded deposit of finished copies. There are two sections devoted to "Encyclopedias and Encyclopedic Reference Works" and "Journals and Newspapers with Mathematical Articles." The appendix gives the "Native American Mathematical Developments" and the "Works Consulted." The "Indices" are ample to meet the needs of the student.

The approximately 3,000 entries of titles and editions are given chronologically. The later editions and issues of each title are listed in order of their publication under the first edition, all of these being collated according to the practices of the Library of Congress. One feature that should be noted is the generous reproduction of title-pages in facsimile. In all there are about 600 of these.

The author makes no undue claims that every mathematical publication in America up through 1850 has been included; however, from a knowledge of the author's pains-taking efforts in other works, we would not hesitate to predict that few new titles will be found. The format of the volume is excellent. The present work should be of great value as a reference work for libraries and collectors as well as a source document for the historian of American mathematics and science.

University of Maryland. A. W. RICHESON.

College Algebra. By H. T. Davis. Prentice-Hall, Inc., New York, 1940. xiii+423 pages. $2.50.

The first seven chapters of this text are given to the elementary topics of factoring, exponents, logarithms, progressions, and linear and quadratic equations. Hence it adapts itself for use by students who have had no more than one year of high school algebra. Beginning with Chapter VIII and continuing through Chapter XVI, the text covers those topics usually given in a college algebra course,—the binomial theorem, the theory of investment, permutations and combinations, proportion and variation, complex numbers, the theory of equations, determinants, inequalities, and infinite series. Included in Chapter XI, on proportion and variation, is an introduction to trigonometry.

The material which appears in the remaining four chapters is, for the most part, not included in many college algebra texts. Chapter XVII gives an introduction to statistics. Among the *Special Topics* of Chapter XVIII, we find continued and partial fractions and the Gregory-Newton formula for interpolation. Under *Mathematical Recreations* in Chapter XIX, appear such topics as scales of notation, criteria for divisibility, prime numbers, magic squares, trisection of an angle, squaring of a circle, and duplication of a cube. The fact that these topics are incorporated as a chapter in a college algebra text should bring them to the attention of more college students and thereby create an interest in them earlier in the student's career. The final chapter, entitled *The meaning of mathematics*, includes among other things a short sketch of the the life of each of a number of mathematicians, as well as some of the advances in science which have been aided by a study of mathematics.

Throughout the text, the author has given considerable attention to the recording of "historical incidents which marked the development of algebra." This should stimulate the interest of the student.

Besides the usual short tables of logarithms, powers and roots, and of those functions used in the chapter on theory of investment, the book contains a short table of sines, cosines, and tangents, one of the logarithms of these functions, a table of primes up to 1000, and a table of coefficients for fitting a straight line.

There are numerous of graded problems to which answers are given. The type is large and clear and the general appearance of the book is attractive.

Wayne University. D. C. MORROW.

College Algebra. By Reagan, Ott, Sigley. Farrar and Rinehart, New York, 1940. 445 pages.

The outstanding feature of this college algebra is its emphasis on the logic of mathematical proof. The first seven chapter headings are: "Fundamental operations, Functions and graphical representation, Fractions and linear equations, Deductive and inductive reasoning, The binomial theorem, Choice, Probability". Exponents are not treated systematically until Chapter 9, page 120. Factoring is not taken up until Chapter 14, page 216. The authors explain in the preface that one can take Chapters 9 and 14 after Chapter 3. (Where else would one take them?)

Nowadays it is unusual to see such a big book devoted to college algebra alone. Engineering students are in a hurry to get on to calculus while most of the others use the unified texts, and take the implied short cuts. Thus the field of Reagan, Ott, Sigley may be pretty well limited to the leisurely liberal arts courses for the better class of students. The book is more reminiscent of the old Wilczynski and Slaught than of any of the modern texts.

Students who do have the time to take a course based on this text will find it carefully written and replete with illustrative examples. Chapters 4, 6, 7 are especially well written although the solution of example 15, p. 71, starts with an unnecessary and unfortunate assumption which would be more suitable anywhere else in the book than it is in the chapter on inductive and deductive reasoning.

Teachers will find that careful thought has been applied to the organization of the text. Unessential sections are starred; the numerous problems are graded in order of increasing difficulty. These and other instances of good editorial work should enhance its teachability.

Tulane University. W. L. DUREN, JR.

College Algebra. By N. J. Lennes. Revised Edition. Harper Brothers, New York, 1940. xii+432 pages. $2.25.

This is so nearly a new book that the copyright page contains the words *first edition.*

There is abundant material for both *slow* and *fast* sections of algebra. The total number of problems and exercises is almost 3300. An important innovation is the placing of important formulas, statements, equations, etc., in *boxes.* Underlines are used for emphasis in place of black type. The page size is the same, but the narrower margins give a more crowded appearance.

The chapter on ratio, proportion, and variation is new. The treatment of logarithms is too brief. There is no mention of some of the common difficulties encountered in the handling of logarithms with negative characteristics. The only mention of natural logarithms is that they are "used for theoretical purposes." Geometric progressions with negative ratios are almost entirely neglected, and are ignored in the discussion of geometric means.

A new chapter is devoted to inequalities. The chapter on the numerical solution of equations includes the finding of integral roots (by synthetic division) and Horner's Method. Unfortunately, practically nothing is said about the finding of rational,

non-integral roots by synthetic division. The chapter on the theory of equations gives a better statement of Descartes' Rule of Signs than is found in most algebra text books. The chapter on infinite series includes practically all of the tests for convergence that are given in the calculus courses, except, of course, the integral test. Chapter 25 consists of forty-six cumulative reviews, each of which, but one, has six problems. A set of three- and four-place tables is supplied, followed by a ten page index.

There is a number of misprints (especially on page 150), but most of them are obvious. There are no answers supplied in the book. The vinculum, or upper bar, is freely used as a sign of aggregation. The non use of X' and Y' on the negative $X-$ and $Y-$axes is to be commended. The figures are well drawn. The better students may find the book too wordy, but the poorer ones will like the amount of detail that is given.

University of Arkansas. EDWIN COMFORT.

An instructor assigned to his algebra class the following problem, among others:

Solve: $$\sqrt{x-9}+\sqrt{x-9}=0.$$

On examining the assignment before the next meeting of the class he mentally kicked himself for having included such a trivial exercise. Imagine his surprise when, on looking over the students' papers, he found that of a class of 21, 7 had "solution" (1), given below, 5 had "solution" (2), 5 had "solution" (3), and 4 had "solution" (4).

(1)
$$\begin{aligned} \sqrt{x-9} &= -\sqrt{x-9} \\ x-9 &= -(x-9) \\ 2x &= 18 \\ x &= 9. \end{aligned}$$

(2)
$$\begin{aligned} x-9+x-9 &= 0 \\ 2x-18 &= 0 \\ x &= 9. \end{aligned}$$

(3)
$$\begin{aligned} \sqrt{x-9} &= -\sqrt{x-9} \\ x-9 &= x-9 \\ x &= 9\ ! \end{aligned}$$

(4)
$$\begin{aligned} x-9+2\sqrt{x-9}\cdot\sqrt{x-9}+x-9 &= 0 \\ 4x-36 &= 0 \\ x &= 9. \end{aligned}$$